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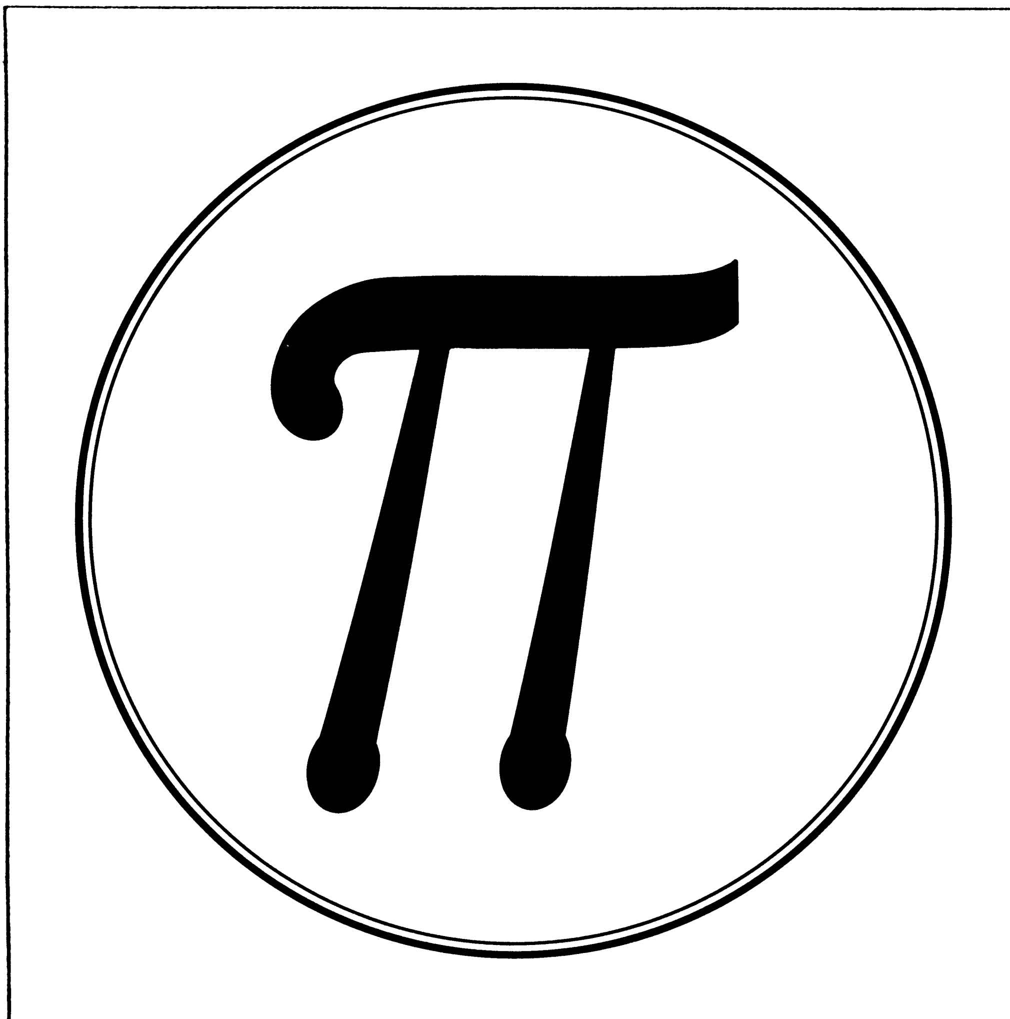

- The Ubiquitous π
- Money Is Irrational
- Secretary's Packet Problem

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AUTHORS

D. Castellanos ("The Ubiquitous Pi") received his Ph.D. from The University of Michigan in 1964. He regularly teaches courses in Complex Analysis and Integral Equations at the School for Graduate Studies at Universidad de Carabobo in Valencia, Venezuela. His main area of research is functional analysis, and he has the history of mathematics as a hobby. In the evolution of man's knowledge of pi he sees a unifying theme in the evolution of mathematics itself. It was the study of this unifying theme that eventually prompted him to write the paper that appears in this issue. In functional analysis his main concern for the past few years has been in spectral analysis of rings of operators associated with spherical representations of groups. He has tried to bring his research to bear on the problem of analyzing the foundations of a dynamical theory and the implications this may have in the study of the physics of elementary processes.

He says that much to the joy of his three daughters and consternation of his calorie-conscious wife, he also has gourmet cooking as a hobby.

Credit: Art for the note by Norwood provided by William Muir

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ARTICLE

The Ubiquitous π

Some well-known and little-known appearances of π in a wide variety of problems.

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Part I

Section 1. Some History

Lost in antiquity is the time when man first became consciously aware of the existence of circles or when he had need to determine their circumference or area. Man certainly saw circles in nature long before the appearance of the wheel. He saw them in the ripples of the water when he dropped a stone into a pond, in the center of a daisy or a sunflower, in the disks of the sun and the moon, or more simply, in the pupils of the eyes of his fellow human beings. Man, though, does not seem to have realized for several millennia, even after the onset of highly organized civilization, that there was a proportional relation between the circumference and the diameter. The first numerical values associated with the circle are believed to be those of the Rhind Papyrus (c. 1650 B.C.). Ahmes the scribe, the writer of the Papyrus, lays down the following rule: *cut off* $1/9$ *of a diameter and construct a square upon the remainder; this has the same area as the circle.* From this construction we can deduce a value of π equal to $(16/9)^2 = 3.16049\ldots$, not very inaccurate [**20**]. But nowhere does the Papyrus make reference to this constant. It was not until the second half of the fifth century before Christ that Hippocrates of Chios made the precise statement that the areas of circles are in the ratio of the square of their diameters, which is equivalent to the statement that the area is a constant times the square of the diameter [**55**]. It was Hippocrates, incidentally, who first gave hope that a square could be constructed with ruler and compass in a finite number of operations which was exactly equal in area to a given circle. Hippocrates had succeeded in obtaining reciprocal relations between areas limited by arcs of a circle and areas limited by straight lines. He called such areas *lunes*, and many of them can be constructed [**8**]. The possibility of *squaring the circle* would not be settled, in the negative, for another twenty three hundred years.

Euclid, in the second half of the fourth century before Christ, proved [**60**] that $3 < \pi < 4$, but it was not until the third century B.C., that Archimedes of Syracuse, the greatest genius of antiquity, attacked this problem systematically. Using polygons inscribed and circumscribed in a circle whose number of sides are successively doubled, so that, for a large number of doublings, they become indistinguishable with the circle, he obtained for π the bounds $3\ 10/71 < \pi < 3\ 1/7$, or in decimal notation $3.140845\ldots < \pi < 3.142857\ldots$. The bound $3\ 1/7 = 22/7$ is often referred to, erroneously, as the Archimedean value. Archimedes meant this as an upper bound on the value of π. Even the great geometer and scholar Michel Chasles fell into this practice.

Archimedes' method remained essentially unchanged, except for better approximations to π obtained by taking larger and larger doublings, until the advent of the calculus. The following values of π are of historical interest:

The Bible [86] (Kings, Chapter VII, v. 23, Chronicles, Chapter IV, v. 2) (c. 550 B.C.) and The Talmud (c. 500 B.C.) both give for π the value 3. *The Almagest* of Claudius Ptolemy (168–128 B.C.) gives $\pi = 3 + 17/120 = 3.14166\ldots$. This relation is equivalent to $(22 + 355)/(7 + 113)$ and so is in-between the Archimedean bound $22/7$ and the value $355/113$ given first by the Chinese Tsu Chung-Chi (5th century), later by Valentin Otho (16th century) and Adriaan Anthoniszoon (17th century) [43].

The Hindu Aryabhata (6th century) gave the value 3.14156 by using the Archimedean method on a polygon of 384 sides.

The Hindu mathematician Brahmagupta (born A.D. 598) was led astray by the fact that the perimeters of polygons of 12, 24, 48, and 96 sides inscribed in a circle with diameter 10 are given by $\sqrt{965}$, $\sqrt{981}$, $\sqrt{986}$, and $\sqrt{987}$. This led him, apparently, to suppose that as more doublings were made the perimeter would approach $\sqrt{1000}$, which would give him the value of $\pi = \sqrt{1000}/10 = \sqrt{10}$ that he believed to be exact.

The Chinese were considerably more advanced in arithmetical manipulation than their western counterparts, for in A.D. 264 Liu Hui using a 192-sided polygon found $3.141024 < \pi < 3.142704$, and with a polygon of 3072 sides the value 3.14159. In the fifth century Tsu Chung-Chi and his son Tsu Keng-Chi found $3.1415926 < \pi < 3.1415927$, bounds not found in the western world until the 16th century [54].

Dante Alighieri is said to have given the estimate $\pi = 3 + \sqrt{2}/10 = 3.1414$ [49]. The Divine Comedy, *Paradiso*, Canto XXXIII, has, towards the end, the curious phrase: *As the geometer who gives himself to squaring the circle, and finds not in his thought the needed principle, so was I before that new image.*

Leonardo of Pisa, Fibonacci, using a 96-sided polygon found the value $864/275 = 3.141818$.

The German astronomer Georg Joachim von Lauchen, called Rhäticus (born in Rhaetia, 1514–1576), published some trigonometric tables with fifteen decimal places [76] from which a highly accurate value of π can be obtained. Since for very small θ, $\sin\theta \simeq \theta$, and half a circumference has 648,000″, then, multiplying $\sin 10''$, whose value is, according to Rhäticus, 0.000048481368092 by 64,800 we obtain $\pi = 3.1415926523$, correct to the eighth decimal place.

π remained essentially in the bounds given above until the coming of the calculus. It is to this discovery that we now turn.

Section 2. The First Analytical Expressions for π

Consider the trigonometric identity,

$$\frac{\sin\theta}{\theta} = \cos(\theta/2)\frac{\sin(\theta/2)}{\theta/2} = \cos(\theta/2)\cos(\theta/4)\frac{\sin(\theta/4)}{\theta/4}$$

$$= \cos(\theta/2)\cos(\theta/4)\cdots\cos(\theta/2^n)\frac{\sin(\theta/2^n)}{\theta/2^n}.$$

As n increases $\sin(\theta/2^n)/(\theta/2^n)$ approaches one and we obtain Euler's formula,

$$\frac{\sin\theta}{\theta} = \cos(\theta/2)\cos(\theta/4)\cos(\theta/8)\cdots. \tag{2-1}$$

If we let $\theta = \pi/2$, (2-1) gives, after we use the formula for the cosine of the half-angle: $\cos(\theta/2) = \sqrt{\frac{1}{2}(1+\cos\theta)}$, the result,

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots, \tag{2-2}$$

first given by François Viète in 1593, who used a slightly different derivation [89]. The convergence of Viète's expression was proved by F. Rudio in 1891 [77].

Viète's formula is the first analytical expression ever obtained for π.

Taking the logarithmic derivative of Euler's formula we obtain,

$$\frac{1}{\theta} = \operatorname{cotan}\theta + (1/2)\tan(\theta/2) + (1/4)\tan(\theta/4) + (1/8)\tan(\theta/8) + \cdots.$$

If we let $\theta = \pi/4$, we obtain,

$$\frac{4}{\pi} = 1 + \frac{1}{2}\tan(\pi/8) + \frac{1}{4}\tan(\pi/16) + \cdots. \tag{2-3}$$

The calculation of π by this formula is equivalent to a geometrical calculation undertaken by René Descartes in the 17th century [22].

Section 3. Euler's Summation Formula

We want now to establish a remarkable summation formula first derived by Euler. We will make use of the symbols $[x]$, for the greatest integer less than or equal to x, and $((x))$ for $x - [x]$.

Euler's summation formula states that if f has a continuous derivative f' on $[a, b]$, then

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)((x))\,dx + f(a)((a)) - f(b)((b)), \tag{3-1}$$

where $\sum_{a<n\leqslant b}$ means the sum from $n = [a]+1$ to $n = [b]$.

If a and b are integers Euler's summation formula (3-1) may be written,

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\left(x - [x] - \frac{1}{2}\right)dx + \frac{1}{2}(f(a) + f(b)). \tag{3-2}$$

This formula is an immediate consequence of the following Riemann-Stieltjes integral identity for integration by parts,

$$\int_a^b f(x)\,d(x-[x]) + \int_a^b (x-[x])\,df(x) = f(b)(b-[b]) - f(a)(a-[a]).$$

Since the greatest-integer function has unit jumps in $[a, b]$ at the integers $x = [a]+1$, $[a]+2,\ldots,[b]$, we can write,

$$\int_a^b f(x)\,d[x] = \sum_{a<n\leqslant b} f(n).$$

If we combine this result with the equation above the formula follows [3].

The function $x-[x]-\frac{1}{2}$, which appears in equation (3-2), is periodic with period 1. If x is in the interval $[0,1]$ this function is simply $x-\frac{1}{2}$. This function has the property, largely responsible for the existence of Euler's summation formula, that its integral in the interval $[0,1]$ is zero. We can, with the same idea in mind, define another function $\phi_2(x)$, such that its derivative is $x-\frac{1}{2}$, which we shall call $\phi_1(x)$, and such that its integral in the interval $[0,1]$ is zero: $\int_0^1\phi_2(x)\,dx=0$. Similarly, $\phi_3'(x)=\phi_2(x)$, and $\int_0^1\phi_3(x)\,dx=0$. In general, we seek a sequence of functions $\phi_n(x)$, $n=1,2,3,\ldots$, such that $\phi_1(x)=x-\frac{1}{2}$, $\phi_n'(x)=\phi_{n-1}(x)$ for $n>1$, and $\int_0^1\phi_n(x)\,dx=0$ for all $n\geqslant 1$. The constant multiples of these functions $n!\phi_n(x)=B_n(x)$ are called, after their discoverer, Bernoulli polynomials. They obey the relation

$$B_n'(x)=nB_{n-1}(x).$$

The first few Bernoulli polynomials are,

$$\begin{aligned}
B_0(x) &= 1,\\
B_1(x) &= x-1/2,\\
B_2(x) &= x^2-x+1/6,\\
B_3(x) &= x^3-(3/2)x^2+(1/2)x,\\
B_4(x) &= x^4-2x^3+x^2-1/30,
\end{aligned}$$

etc.

It is clear from the construction that $B_n(x)$ is a polynomial of degree n. They are defined in the interval $0\leqslant x\leqslant 1$. Their periodic continuations outside this interval are called Bernoulli functions $\overline{B}_n(x)$. Notice that the $\overline{B}_n(x)$ are continuous if n is greater than or equal to two, since we have,

$$B_n(1)-B_n(0)=\int_0^1 B_n'(x)\,dx=n\int_0^1 B_{n-1}(x)\,dx=0, \qquad n\geqslant 2,$$

by the definition of $B_1(x)$, $B_2(x)$, etc.

It is easy to prove that $\overline{B}_n(x)$ is an odd function for odd n and an even function for even n.

The constant terms of the Bernoulli polynomials form a particularly interesting set of numbers. We set $B_n=B_n(0)$. It is clear from the way the polynomials $B_n(x)$ are constructed that all the B_n are rational numbers. It can be shown that $B_{2n+1}=0$ for $n\geqslant 1$, and is alternately positive and negative for even n. The B_n are called Bernoulli numbers, and the first few are:

$$\begin{matrix} B_0=1, & B_1=-1/2, & B_2=1/6, & B_4=-1/30, & B_6=1/42,\\ B_8=-1/30, & B_{10}=5/66, & B_{12}=-691/2730, & B_{14}=7/6, & \end{matrix} \tag{3-3}$$

etc.

Bernoulli polynomials and numbers are intimately related to the Maclaurin expansions of some of the elementary functions. To exhibit these relations let us find a generating function for Bernoulli polynomials,

$$F(x,t)=\sum_{n=0}^{\infty}\frac{B_n(x)t^n}{n!}. \tag{3-4}$$

This series converges uniformly for all $|t|<2\pi$ and for all x. We may, therefore, differentiate term-by-term with respect to x:

$$\frac{\partial F(x,t)}{\partial x}=\sum_{n=0}^{\infty}\frac{B_{n-1}(x)t^n}{(n-1)!}=t\sum_{n=0}^{\infty}\frac{B_n(x)t^n}{n!}=tF(x,t).$$

Thus $F(x,t)$ satisfies the differential equation $\frac{\partial F}{\partial x}=tF$, whose general solution is $F(x,t)=T(t)e^{xt}$, with T an arbitrary function of t. To determine T we integrate series (3-4) with respect to x between 0 and 1,

$$\int_0^1 F(x,t)\,dx=T(t)\int_0^1 e^{xt}\,dx=T(t)(e^t-1)/t$$
$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^1 B_n(x)\,dx=1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\int_0^1 B_n(x)\,dx=1.$$

Hence $T(t)=t/(e^t-1)$, and (3-4) becomes,

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(x)t^n}{n!}. \tag{3-5}$$

Bernoulli polynomials are often defined by means of (3-5).

If we substitute $x=0$ in (3-5), add $t/2$ to both sides, remembering that $B_0=1$, $B_1=-\frac{1}{2}$, and that $B_n=0$ for *odd* n greater than or equal to 3, we obtain

$$\frac{t}{2}\coth\frac{t}{2}=\sum_{n=0}^{\infty}\frac{B_{2n}t^{2n}}{(2n)!}.$$

This expression is valid for all $|t|<2\pi$. Changing t to $2ix$ we have,

$$x\operatorname{cotan}x=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}B_{2n}x^{2n}}{(2n)!}. \tag{3-6}$$

This formula will be very useful to us in the next section in summing the reciprocals of the even powers of the integers.

Now that we have derived the most important properties of the Bernoulli polynomials let us return to Euler's summation formula. Let, in (3-2), $f(x)=1/x$, $a=1$, $b=n$, to obtain

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\int_1^n\frac{dx}{x}-\int_1^n\frac{\overline{B}_1(x)\,dx}{x^2}+\frac{1}{2}\left(1+\frac{1}{n}\right)$$
$$=\log n+\frac{1}{2}+\frac{1}{2n}-\int_1^n\frac{\overline{B}_1(x)\,dx}{x^2},$$

which we may rewrite as

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n=\frac{1}{2}+\frac{1}{2n}-\int_1^n\frac{\overline{B}_1(x)\,dx}{x^2}.$$

If in this formula we let $n\to\infty$, the integral on the right converges since $|\overline{B}_1(x)|\leqslant\frac{1}{2}$

for all x, so that the integral is less in absolute value than that of the convergent integral $\int_1^\infty dx/x^2$. Thus,

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k}-\log n\right]=\frac{1}{2}-\int_1^\infty\frac{\bar{B}_1(x)}{x^2}\,dx=\gamma,$$

a constant called the Euler-Mascheroni constant, which is approximately equal to 0.57721 56649 01532 86060.... It is not known whether γ is rational or not.

The Euler-Mascheroni constant is, incidentally, related to π through the following expression [47],

$$\gamma=\log\frac{4}{\pi}-\sum_{n=1}^{\infty}\frac{(-1)^n S_{n+1}}{2^n(n+1)},$$

where,

$$S_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots. \tag{3-7}$$

Frederick W. Odena [58] found the following curious approximation of the Euler-Mascheroni constant,

$$(0.11111111)^{1/4}=0.577350\ldots,$$

correct to three decimal places.

Other approximations are:

$$(7/83)^{2/9}=0.57721521\ldots, \tag{3-8}$$

correct to six decimal places;

$$\left(\frac{520^2+22}{52^4}\right)^{1/6}=0.5772156634\ldots, \tag{3-9}$$

correct to eight decimal places;

$$\left(\frac{80^3+92}{61^4}\right)^{1/6}=0.57721566457\ldots, \tag{3-10}$$

correct to nine decimal places; and

$$\frac{990^3-55^3-79^2-4^2}{70^5}=0.5772156649015295\ldots, \tag{3-11}$$

less than γ by about 10^{-14}.

Formulas (3-8) through (3-11) are due to the author.

Consider now the function $f(x)=1/(x^2+n^2)$ and let, in Euler's summation formula (3-2), $a=0$, $b=n$, to obtain,

$$\frac{1}{n^2}+\frac{1}{n^2+1^2}+\cdots+\frac{1}{n^2+n^2}=\int_0^n\frac{dx}{n^2+x^2}-\int_0^n\frac{2x\bar{B}_1(x)}{(n^2+x^2)^2}\,dx+\frac{1}{2}\left(\frac{1}{2n^2}+\frac{1}{n^2}\right)$$
$$=\frac{\pi}{4n}+\frac{3}{4n^2}-\int_0^n\frac{2x\bar{B}_1(x)}{(n^2+x^2)^2}\,dx.$$

It can be shown that for large n the last integral behaves like $1/24n^3$. We obtain, hence, the interesting formula, due to Euler,

$$\pi = \lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{6n^2} + 4n\left(\frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \cdots + \frac{1}{n^2+n^2}\right)\right]. \quad (3\text{-}12)$$

Euler communicated this formula in a letter to Christian Goldbach [23].

The values of this sequence for several values of n are:

n	term of the sequence
1	3.166666666666667
2	3.141666666666667
3	3.141595441595442
4	3.141593137254902
5	3.141592780477657
10	3.141592655573826
20	3.141592653620795
30	3.141592653592515
50	3.141592653589920
100	3.141592653589795
112	3.141592653589793

It is seen that beyond n equal 10 the convergence is quite slow. This formula is not appropriate for a high precision calculation of π.

Euler found also the expression [16]:

$$\pi = \frac{1}{n} + 4n\left(\frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \cdots + \frac{1}{n^2+n^2}\right) -$$
$$\frac{4\pi}{e^{2\pi n}-1} + \frac{B_2}{1\cdot n^2} - \frac{B_6}{3\cdot 2^2\cdot n^6} + \frac{B_{10}}{5\cdot 2^4\cdot n^{10}} - \frac{B_{14}}{7\cdot 2^6\cdot n^{14}} + \cdots. \quad (3\text{-}13)$$

For a given accuracy one can always choose n such that the term involving the exponential above becomes negligible. For example, for $n = 6$, $4\pi/(e^{12\pi}-1)$ is $5.33\ldots\times 10^{-16}$. Estimating the exponential term, and using $n = 5$, Euler calculated π to 15 decimals [16].

Section 4. Leonhard Euler and the Question of Jacques Bernoulli

Jacques Bernoulli, the Swiss mathematician who discovered the polynomials we studied in the last section, was a brilliant individual, a contemporary of Newton and Leibniz. He discovered the sum of many important infinite series. He was vexed, however, by the determination of the sum of the reciprocals of the squares of the natural numbers. *If somebody should succeed,*—wrote Bernoulli,—*in finding what till now withstood our efforts and communicate it to us, we shall be much obliged to him* [61].

This problem came to the attention of Euler, who solved it by the following bold line of reasoning:

Euler considered the function,

$$f(x) = \begin{cases} 1, & x = 0 \\ \dfrac{\sin x}{x}, & x \neq 0 \end{cases},$$

which is continuous everywhere and vanishes at the points $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Euler assumed, hence, that $f(x)$ could be written as

$$f(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right), \tag{4-1}$$

since it equals 1 when $x = 0$, and vanishes at $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

We know today, and shall prove it in a moment, that expression (4-1) is indeed correct, but at the time Euler announced it, it was simply a daring conjecture.

It can be shown, from the general theory of infinite products, that if

$$P = \prod_{n=1}^{\infty} \frac{(n+a_1)(n+a_2)\cdots(n+a_s)}{(n+b_1)(n+b_2)\cdots(n+b_s)}, \tag{4-2}$$

and none of the a's or b's is a negative integer, and furthermore

$$\sum_{k=1}^{s} a_k = \sum_{k=1}^{s} b_k, \tag{4-3}$$

then,

$$P = \prod_{k=1}^{s} \frac{\Gamma(1+b_k)}{\Gamma(1+a_k)}, \tag{4-4}$$

where $\Gamma(z)$ is the Gamma function of z [**65**]. Writing Euler's product (4-1) in the form,

$$f(x) = \prod_{n=1}^{\infty} \frac{\left(n - \frac{x}{\pi}\right)\left(n + \frac{x}{\pi}\right)}{n \cdot n}, \tag{4-5}$$

we see that (4-3) is satisfied. If use is made of the results [**65**],

$$\Gamma(1+z) = z\Gamma(z), \tag{4-6}$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{4-7}$$

we obtain from (4-4) and (4-5), $f(x) = \sin x/x$, $x \neq 0$, as Euler surmised.

Euler proceeded to multiply out the factors in (4-1) and to collect powers of x, to obtain,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + \cdots. \tag{4-8}$$

On the other hand, from the Maclaurin expansion for $\sin x/x$, Euler knew that,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots. \tag{4-9}$$

On comparing the coefficients of x^2 on (4-8) and (4-9) Euler found,

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots, \tag{4-10}$$

the sum of the series which defeated Jacques Bernoulli.

Many other proofs of this result have since been found, some of them quite elementary. In order of appearance, the following papers treat this, so called, *Basler Problem*, [91], [92], [52], [82], [35], [83], [27], [59]. Of these, [91], [92], [35], and [59], achieve this result using only de Moivre's identity, or elementary trigonometric results easily derived from it.

Evidence of the wide appeal of this problem is that Euler's formula (4-10) has been explicitly reproduced in the Encyclopedia Britannica [94].

Euler went on to compare the coefficients of the fourth power of x, and found the sum of the series,

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots. \tag{4-11}$$

Euler considered also the expression,

$$1 - \sin x = 0, \tag{4-12}$$

which has the roots: $\pi/2,\ -3\pi/2,\ 5\pi/2,\ -7\pi/2, \ldots$. However, each of these roots is a double root, that is, the curve $y = 1 - \sin x$ is tangent to the x-axis at these points, or, equivalently, the first derivative of the function vanishes along with the function at each of these points. We can, hence, write

$$\begin{aligned} 1 - \sin x &= 1 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \\ &= \left(1 - \frac{2x}{\pi}\right)^2\left(1 + \frac{2x}{3\pi}\right)^2\left(1 - \frac{2x}{5\pi}\right)^2\left(1 + \frac{2x}{7\pi}\right)^2 \cdots. \end{aligned}$$

Comparing the linear terms, Euler found,

$$-1 = -\frac{4}{\pi} + \frac{4}{3\pi} - \frac{4}{5\pi} + \frac{4}{7\pi} - \cdots,$$

or,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{4-13}$$

And Euler was led by his method to a series whose sum was known (Leibniz' series). This convinced him that his method could be trusted, for he wrote: *For our method, which may appear to some as not reliable enough, a great confirmation comes here to light. Therefore, we should not doubt at all of the other things which are derived by the same method* [25].

Euler went on to obtain more from equation (4-1). For instance, letting $x = \pi/2$ in (4-1), one obtains,

$$\begin{aligned} \frac{2}{\pi} &= \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{16}\right)\left(1 - \frac{1}{36}\right)\left(1 - \frac{1}{64}\right) \cdots \\ &= \frac{1\cdot 3}{2\cdot 2} \cdot \frac{3\cdot 5}{4\cdot 4} \cdot \frac{5\cdot 7}{6\cdot 6} \cdot \frac{7\cdot 9}{8\cdot 8} \cdot \frac{9\cdot 11}{10\cdot 10} \cdots, \end{aligned}$$

which written in the more familiar form,

$$\pi = 2 \lim_{n\to\infty} \left(\frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots (2n-1)}\right)^2, \tag{4-14}$$

is recognized as Wallis' product, which was well known to Euler. John Wallis' derivation of this formula bears no resemblance to Euler's method [85].

Wallis' formula may also be written as [44]:

$$\pi = 2 \lim_{n\to\infty} \left[2^{2n} \Big/ \binom{2n}{n} \right]^2, \tag{4-15}$$

where $\binom{2n}{n}$ is the binomial coefficient.

Let us now take the logarithmic derivative of (4-1), to obtain,

$$\operatorname{cotan} x - \frac{1}{x} = \sum_{n=1}^{\infty} \left(\frac{1}{x+n\pi} + \frac{1}{x-n\pi} \right) = \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2\pi^2},$$

or,

$$x \operatorname{cotan} x = 1 - 2x^2 \sum_{n=1}^{\infty} \left(\frac{1}{n^2\pi^2} + \frac{x^2}{n^4\pi^4} + \frac{x^4}{n^6\pi^6} + \cdots \right). \tag{4-16}$$

Comparing this result with equation (3-6), and equating equal powers of the variable, we obtain at once the remarkable result,

$$S_{2n} = \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \cdots = \frac{(-1)^{n+1} 2^{2n} B_{2n} \pi^{2n}}{2(2n)!}, \tag{4-17}$$

from which we can obtain:

$$S_2 = \pi^2/6, \qquad S_4 = \pi^4/90, \qquad S_6 = \pi^6/945,$$
$$S_8 = \pi^8/9450, \ \ldots, \ S_{26} = 2^{24} 76{,}977{,}927 \pi^{26}/27!,$$

etc.

Formula (4-17) is, itself, a special case of the more general formula [3],

$$\bar{B}_{2n}(x) = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{k^{2n}}. \tag{4-18}$$

This formula corresponds to the Fourier expansion of the periodic extension $\bar{B}_{2n}(x)$ of the Bernoulli polynomials $B_{2n}(x)$.

Riemann's zeta function $\zeta(s)$ is defined by,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad (s > 1). \tag{4-19}$$

It can be shown [3] that,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \qquad s > 1. \tag{4-20}$$

Hence, for example, with $s = 2$, we have,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots = (1 - 2^{-1})\zeta(2) = \frac{\pi^2}{12}. \tag{4-21}$$

One can, by means of Fourier series, show that [90]

$$\frac{\pi x(\pi - x)}{8} = \sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots \qquad (0 \leqslant x \leqslant \pi).$$

With $x = \pi/2$, this gives

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots. \tag{4-22}$$

This result is a special case of the general formula [3], with $n = 1$, and $x = \frac{1}{4}$,

$$\bar{B}_{2n+1}(x) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{k^{2n+1}}. \tag{4-23}$$

From the equation [**66**]

$$\prod_{p=2}^{\infty} \left(1 - \frac{1}{p^n}\right) = \frac{1}{S_n} = \frac{1}{\zeta(n)}, \tag{4-24}$$

where p are the prime numbers, it is relatively easy to obtain the expression [**66**], [**45**]:

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} + \cdots = \frac{S_n^2 - S_{2n}}{2S_n}, \tag{4-25}$$

where the numbers 2, 3, 5, 7, 8, ..., contain an *odd* number of prime divisors. Special cases of this formula are

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \cdots = \frac{\pi^2}{20},$$

$$\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \cdots = \frac{\pi^4}{1260}.$$

One can find as well,

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n} + \frac{1}{23^n} + \frac{1}{29^n} + \frac{1}{30^n} + \frac{1}{31^n} + \cdots = \frac{S_n^2 - S_{2n}}{2S_nS_{2n}}, \tag{4-26}$$

where 2, 3, 5, 7, 11, ..., are numbers containing an *odd* number of *dissimilar* prime divisors. Examples of this formula are,

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{9}{2\pi^2},$$

$$\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{15}{2\pi^4}.$$

There is also the formula,

$$\frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{12^n} + \cdots = \frac{S_n(S_{2n} - 1)}{S_{2n}}, \tag{4-27}$$

where 4, 8, 9, 12, ..., are composite numbers having at *least* two equal prime divisors.

There exist integral representations for Riemann's zeta function $\zeta(s)$, and it is therefore, in principle, possible to represent the sum of the reciprocals of the *odd* powers of the natural numbers as an integral [**44**], [**64**]. For this case, though, the available results are of the type [**42**], [**14**]:

$$\zeta(3)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}, \tag{4-28}$$

or [84],

$$\zeta(2s+1)=\left(\frac{1}{2}\pi\right)^{2s+1}\lim_{m\to\infty}\frac{1}{m^{2s+1}}\sum_{n=1}^{m}\left(\operatorname{cotan}\frac{n}{2m+1}\right)^{2s+1}. \tag{4-29}$$

No simple algebraic results exist. See, however, T. M. Apostol, Another elementary proof of Euler's formula for $\zeta(2n)$, *American Mathematical Monthly*, 80 (1973), 425–431.

Section 5. Approximations to π and Some Curious Results

It is not difficult to obtain formulas that approximate π. For instance, from the series,

$$\sin x = x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots,$$

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots,$$

$$\tfrac{1}{2}\sin 2x=\sin x\cos x = x-\frac{2}{3}x^3+\frac{2}{15}x^5-\cdots,$$

we obtain, considering the first two equations, and neglecting terms of order five or higher,

$$3\sin x - x\cos x \approx 2x,$$

so that

$$x\approx\frac{3\sin x}{2+\cos x}, \tag{5-1}$$

formula given by Cardinal Nicolaus Cusanus (1401–1464), and later by the Dutch mathematician and physicist Willebrord Snellius (1580–1626) [39].

Taking into account all three of the above series, we obtain, neglecting terms of order seven or more,

$$14\sin x - 6x\cos x + \sin x\cos x \approx 9x,$$

or,

$$x\approx \sin x\frac{14+\cos x}{9+6\cos x}, \tag{5-2}$$

a formula given by Newton.

By slightly modifying Newton's procedure, I recently found the formula

$$x\approx\sin x\frac{187+24\cos x-\cos 2x}{120+90\cos x}.$$

These approximations are the better, the smaller the angle. For an angle of fifteen degrees ($\pi/12$), Cardinal Cusanus' formula gives $\pi=3.141509994$, Newton's formula gives $\pi=3.141592169$, while my formula gives $\pi=3.141592648$; the true value being, of course, $\pi=3.141592653589\ldots$.

The most popular approximation to π is, because of its simplicity,

$$\frac{355}{113} = 3.141592920, \tag{5-3}$$

correct to the sixth decimal place. This fraction may be obtained by a sort of numerological hocus-pocus. Write the first three odd integers in pairs: 1, 1, 3, 3, 5, 5; then put the last three above the first three to make the fraction 355/113.

Other interesting approximations to π are:

$$2 + \sqrt{1 + (413/750)^2} = 3.1415926497, \tag{5-4}$$

$$1.09999901 \times 1.19999911 \times 1.39999931 \times 1.69999961 = 3.141592573. \tag{5-5}$$

Formula (5-4) and other less accurate ones were published in *L'Intermédiaire des Mathématiciens* [**50**]. Formula (5-5) is a simple modification of a result known for some time. Each of the factors in (5-5) has five nines, and each is symmetrical about the middle nine.

Ramanujan noticed that if the fourth power of π (97.40909108) is subtracted from 97.5, the result (0.090908920) is very nearly the reciprocal of 11 (0.090909091). Hence $(97.5 - (1/11))^{1/4}$ should be a very good approximation to π. When cleared of decimals this approximation can be written,

$$\left(9^2 + \frac{19^2}{22}\right)^{1/4} = 3.14159265258\ldots. \tag{5-6}$$

This approximation may also be written in the very appealing form,

$$\left(102 - \frac{2222}{22^2}\right)^{1/4} = 3.14159265258\ldots. \tag{5-7}$$

Imitating Ramanujan's method, noticing that the fifth power of π (306.0196848) is nearly $306 + (1/50.8) = 306.0196850$, the author found,

$$\left(\frac{77729}{254}\right)^{1/5} = 3.1415926541\ldots. \tag{5-8}$$

Inexplicably, Ramanujan seems to have missed this simple result.

Ramanujan obtained also the remarkable formulas,

$$\frac{63}{25}\left(\frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}}\right) = 3.14159265380\ldots, \tag{5-9}$$

and

$$\frac{12}{\sqrt{130}}\log\left\{\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right\}, \tag{5-10}$$

$$\frac{12}{\sqrt{310}}\log\left\{\frac{1}{4}(3+\sqrt{5})(2+\sqrt{2})\left[(5+2\sqrt{10}) + \sqrt{(61+20\sqrt{10})}\right]\right\}, \tag{5-11}$$

$$\frac{4}{\sqrt{522}}\log\left\{\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^3(5\sqrt{29}+11\sqrt{6})\left[\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right]^6\right\}. \tag{5-12}$$

In these last three formulas the logarithm is the natural logarithm. They are correct to 15, 22, and 31 decimal places, respectively [**67**].

Ramanujan's method of approximating π is based on a discovery of Hermite that for certain integral values of n, $e^{\pi\sqrt{n}}$ is nearly an integer. See Charles Hermite, *Sur la théorie des équations modulaires*, Comptes Rendus, 16 May, 13 June, 20 June, 4 July, 18 July, and 25 July, 1859, and L. Kronecker, *Berliner Monatsberichte*, 1863, pp. 322–375. For instance, $e^{\pi\sqrt{58}} = 24591257751.99999982\ldots$, $e^{\pi\sqrt{163}} = 262{,}537{,}412{,}640{,}768{,}743.99999999999925\ldots$.

Ramanujan also gave the approximation,

$$\frac{355}{113}\left(1 - \frac{0.0003}{3533}\right) = 3.1415926535897943,$$

greater than π by about 10^{-15}.

Other approximations to π are:

$$\frac{47^3 + 20^3}{30^3} - 1 = 3.141592593\ldots; \tag{5-13}$$

$$\left(31 + \frac{62^2 + 14}{28^4}\right)^{1/3} = 3.14159265363\ldots, \tag{5-14}$$

greater than π by about 10^{-10};

$$\frac{1700^3 + 82^3 - 10^3 - 9^3 - 6^3 - 3^3}{69^5} = 3.1415926535881\ldots, \tag{5-15}$$

less than π by about 10^{-12}; and

$$\left(100 - \frac{2125^3 + 214^3 + 30^3 + 37^2}{82^5}\right)^{1/4} = 3.141592653589780\ldots, \tag{5-16}$$

less than π by about 10^{-14}.

Formulas (5-13) through (5-16) are due to the author.

There exist many geometrical constructions where algebraic approximations to π are used to obtain approximate squarings of the circle. Grunert gave a geometrical construction for π based on the fact that $355/113 = 3 + 4^2/(7^2 + 8^2)$ [**30**]. Ramanujan gave a geometrical construction of the value (5-6) which for a linear segment equal to one sixth of $2\pi r$ has an error less than a twelfth of an inch when the diameter of the circle is 8000 miles long! [**67**].

G. Stanley Smith discovered that 553/312 is a good approximation of the square root of π, $1.7724538509\ldots$, giving it correctly to the fourth decimal place. It is remarkable that this fraction is nearly 355/113 written backwards:

$$\frac{553}{311 + 1} = 1.772435897\ldots. \tag{5-17}$$

Another curious approximation of the square root of π, which involves, apart from the exponent, all the digits of the rounded-off value 3.141593, is:

$$\left(\frac{3}{14}\right)^2 \frac{193}{5} = 1.772448980\ldots. \tag{5-18}$$

A slightly better approximation is:

$$\frac{66^3 + 86^2}{55^3} = 1.772453794\ldots. \tag{5-19}$$

Also

$$\frac{296}{167} = 1.77245509\ldots.$$

The last three results are due to the author.

A very surprising property of π was discovered by T. E. Lobeck of Minneapolis [**26**]. Starting with the conventional 5-by-5 magic square shown below, and then substituting the nth digit of π for each number n in the square, we obtain a new array of numbers. The sum of the numbers in every column is duplicated by a sum of numbers in every row!

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Five-by-five magic square

2	4	3	6	9	(24)
6	5	2	7	3	(23)
1	9	9	4	2	(25)
3	8	8	6	4	(29)
5	3	3	1	5	(17)
(17)	(29)	(25)	(24)	(23)	

Magic square modified by π

Martin Gardner mentioned the following curiosity, discovered by James Davis, in a private communication to the author:

Write the letters of the English alphabet, in capitals, clockwise around a circle, and cross-out the letters that have right-left symmetry, A, H, I, M, etc. The letters that remain group themselves in sets of 3, 1, 4, 1, 6.

R. G. Duggleby, a biochemist at the University of Ottawa, discovered the unexpected result [**26**],

$$(\pi^4 + \pi^5)^{1/6} = 2.718281809\ldots, \tag{5-20}$$

the number e, correct to the seventh decimal place! Martin Gardner said of formulas such as this that they show that "you can have your *pi* and *e*at it too."

With some patience, any mathematical constant can be approximated by an interesting combination of another mathematical constant. For example,

$$\frac{\pi^9 + \pi^5 - (27/7)\pi}{10^5} = 0.3010300145\ldots$$

differs from the decimal logarithm of 2 by 1.88×10^{-8}.

The first ten decimal places of the tail of e have a sort of repetitive pattern: eights alternate with eights, and ones alternate with twos. So if we multiply e by 1000, and e by 10, and add the two results we make each eight in the first summand coincide with an eight in the second summand, each one in the first summand coincide with a two in the second summand, and each two in the first summand coincide with a one in the second summand and we get nearly a repeating decimal. Indeed:

$$\begin{array}{r} 2718.281828459\,+ \\ \underline{27.182818284} \\ 2745.464646743 \end{array}$$

Since $\frac{46}{99} = 0.4646464646\ldots$, then

$$\frac{2745+\frac{46}{99}}{1010}=\frac{271801}{99990}=2.7182818281\ldots$$

is a good approximation to e less than the true value by about 3×10^{-10}.

This approximation confirms a belief held by the author that if you stare at a number long enough the number will talk back to you!

Other approximations to e are:

$$2+\frac{54^2+41^2}{80^2}=2.718281250\ldots, \tag{5-21}$$

correct to six decimal places;

$$\left(150-\frac{87^3+12^5}{83^3}\right)^{1/5}=2.718281828435\ldots, \tag{5-22}$$

correct to ten decimal places;

$$4-\frac{300^4-100^4-1291^2+9^2}{91^5}=2.718281828459609\ldots, \tag{5-23}$$

correct to twelve decimal places; and

$$\left(1097-\frac{55^5+311^3-11^3}{68^5}\right)^{1/7}=2.7182818284590 46\ldots, \tag{5-24}$$

greater than e by about 10^{-15}.

Formulas (3-8) through (3-11), (5-13) through (5-16), (5-19), and (5-21) through (5-24) above, were found by the author with a computer program based on the formula,

$$N^{\alpha}|s^{\beta}-[s^{\beta}]+m|=M, \tag{5-25}$$

where N took integer values between 2 and 99, α and β took positive integer values not larger than 9, and m the values 0, ± 1, ± 2, ± 3, ± 4, and ± 5. The brackets, as before, designate the integral part of s^{β} and s is the number one wishes to approximate. The computer was asked to print those M's, if any, that differed from an integer by not more than ± 0.01. These M's were then approximated by the nearest integer $\overline{M}$, and the $\overline{M}$'s were decomposed as an algebraic sum of powers of integers. Then s was found to be

$$s=\left([s^{\beta}]-m\pm\frac{\overline{M}}{N^{\alpha}}\right)^{1/\beta}, \tag{5-26}$$

where the signs must be chosen so that the quantity in parentheses is positive.

This simple algorithm is surprisingly effective. The run for γ, the Euler-Mascheroni constant, for instance, yielded 583 approximations with six decimals or more!

The decomposition of the $\overline{M}$'s into algebraic sums of powers of integers was done by the computer in an essentially trial-and-error basis. This method would be considerably enhanced if an algorithm could be found for decomposing a positive integer N in the form

$$N=x_1^{\alpha_1}\pm x_2^{\alpha_2}\pm\cdots\pm x_i^{\alpha_i},$$

for some i. The α's are not necessarily equal, and the x's are restricted to be two or three-digit numbers. It is this last requirement that complicates the problem!

It is known, for instance, that every positive integer is the sum of four squares. If one uses this information on the result,

$$75^4(\pi^4 - 95) \approx 76{,}225{,}146,$$

one finds

$$76{,}225{,}146 = 2 \cdot 3 \cdot 127 \cdot 167 \cdot 599 = 8585^2 + 1364^2 + 579^2 + 572^2.$$

The computer found the much neater result

$$76{,}225{,}146 = 93^4 + 34^4 + 17^4 + 88,$$

which gives the interesting formula

$$\left(95 + \frac{93^4 + 34^4 + 17^4 + 88}{75^4}\right)^{1/4} = 3.141592653590\ldots, \tag{5-27}$$

greater than π by about 10^{-12}.

A sort of reciprocal relation of (5-20) was discovered by the author

$$(2e^3 + e^8)^{1/7} = 3.141716839\ldots, \tag{5-28}$$

greater than π by about 10^{-4}.

We can improve the approximation by slightly decreasing the exponent:

$$(2e^3 + e^8)^{1/7.0001} = 3.141665461\ldots. \tag{5-29}$$

We would like, however, to get rid of the digits in the tail of (5-29) and replace them by the correct digits of π. We can achieve this by adding the decimal part of (5-28) to the 1 in the exponent of (5-29):

$$(2e^3 + e^8)^{1/(7.0001 + 0.000141716839)} = 3.141592653\ldots! \tag{5-30}$$

Does the reader see why this works?

Section 6. Two Series for the Arctangent

If we expand $1/(1 + x^2)$ in powers of x and integrate term-by-term we obtain

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}. \tag{6-1}$$

This expansion converges if $|x| < 1$.

Series (6-1) was discovered by James Gregory in 1671. It converges very slowly except for very small values of its argument. For $x = 1$ it yields Leibniz' celebrated series for $\pi/4$, (4-13), which requires two thousand terms to give three decimal figures of π.

Let us use Pochhammer's symbol $(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1)$, $\alpha \neq 0$, and the identity $(3/2)_n/(1/2)_n = 2n + 1$ to write (6-1) in hypergeometric form:

$$\tan^{-1}x = xF(1, 1/2; 3/2; -x^2). \tag{6-2}$$

Consider now the relation

$$F(a,b;c;z) = (1-z)^{-a} F(a, c-b; c; -z/(1-z)), \tag{6-3}$$

valid if $|z| < 1$, and $|z/(1-z)| < 1$. This relation is an equality among two of Kummer's twenty-four solutions to Gauss' hypergeometric differential equation. In (6-3) let $a = 1$, $b = 1/2$, $c = 3/2$, and $z = -x^2$, to obtain

$$\tan^{-1} x = xF(1,1/2;3/2;-x^2) = [x/(1+x^2)]F(1,1;3/2;x^2/(1+x^2)).$$

Since $(2n+1)! = (2)_{2n} = 2^{2n} n!(3/2)_n$, the above equation gives

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}}. \tag{6-4}$$

Inasmuch as $x^2/(1+x^2) < 1$ for every real x, we conclude that (6-4) converges for every real value of its argument.

Equation (6-4) is Euler's famous series for the arctangent, discovered in 1755. This series converges very rapidly for all x, and especially for small values of its argument.

Section 7. Three Series for the Arctangent

In the year 1202 the Italian mathematician Leonardo of Pisa, better known by the nickname of *Fibonacci*, a short form of *Filius Bonacci* meaning *Son of Bonacci*, published a book with the title *Liber Abaci*. It is in this book that the Fibonacci sequence appears for the first time.

Fibonacci did not discover any of the properties of the sequence which bears his name. He simply proposed and solved, in the *Liber Abaci*, the problem of determining how many rabbits would be born in one year starting from a given pair. With some natural assumptions about their breeding habits, the population of rabbit pairs per month corresponds to the elements of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13,..., where beginning with 0 and 1, each term of the sequence is the sum of the two preceding ones.

With the passage of time this sequence would appear in so many areas with no possible connection with the breeding of rabbits that in 1877 *Edouard Lucas* proposed naming it the *Fibonacci Sequence* and its terms *Fibonacci Numbers*. The fertility of this sequence seems to be inexhaustible, and every year new and curious properties of it are discovered. The standard symbol for Fibonacci numbers is F_n, and the defining relation is

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \qquad F_1 = 1.$$

Binet's formula [**36**] for Fibonacci numbers is:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{7-1}$$

Consider the identity

$$\tan^{-1}\frac{\sqrt{5}\,x}{1-x^2} = \tan^{-1}\left(\frac{1+\sqrt{5}}{2}\right)x - \tan^{-1}\left(\frac{1-\sqrt{5}}{2}\right)x, \tag{7-2}$$

easily verified by taking the tangent of both sides.

Let us substitute (6-1) in the two terms on the right-hand side above, and make use of equation (7-1) to obtain, after changing $\sqrt{5}\,x$ by t,

$$\tan^{-1}\frac{5t}{5-t^2}=\sum_{n=0}^{\infty}\frac{(-1)^nF_{2n+1}t^{2n+1}}{5^n(2n+1)}.$$

If we now let $5t/(5-t^2)=\alpha>0$, and choose the smaller of the roots of this quadratic equation, we obtain

$$\tan^{-1}\alpha=\sum_{n=0}^{\infty}\frac{(-1)^nF_{2n+1}t^{2n+1}}{5^n(2n+1)}, \tag{7-3}$$

with

$$t=\frac{2\alpha}{1+\sqrt{1+(4\alpha^2/5)}}, \tag{7-4}$$

a curious and simple series for the arctangent with odd Fibonacci numbers as coefficients.

Iteration of the process used to obtain equation (7-3) leads to an additional series for the arctangent:

$$\tan^{-1}\alpha=5\sum_{n=0}^{\infty}\frac{(-1)^nF_{2n+1}^2}{(2n+1)\left(t+\sqrt{t^2+1}\right)^{2n+1}}, \tag{7-5}$$

with

$$t=\frac{5}{4\alpha}\left(1+\sqrt{1+(24\alpha^2/25)}\right). \tag{7-6}$$

One more iteration gives

$$\tan^{-1}\alpha=\sum_{n=0}^{\infty}\frac{(-1)^n5^{n+2}F_{2n+1}^3}{(2n+1)\left(t+\sqrt{t^2+5}\right)^{2n+1}}, \tag{7-7}$$

with t the largest positive root of

$$8\alpha t^4-100t^3-450\alpha t^2+875t+625\alpha=0. \tag{7-8}$$

This quartic equation is, in principle, solvable by radicals for any value of α. The algorithm, though, does not seem to lead to any manageable combination of radicals, and for its solution it is better to resort to Newton's iterative procedure. We will discuss several solutions later on.

These arctangent series converge very rapidly, and the convergence improves with each iteration. All three of them converge faster than Gregory's or Euler's series (6-1) and (6-4). Formulas (7-5)–(7-6) are especially amenable for numerical work. These series were discovered by the author. Details regarding convergence are given in reference [**17**].

Section 8. Some Analytical Expressions for π

There exists an *embarras de richesses* of analytical formulas that yield π or expressions related to π. One of the most beautiful ones is

$$\frac{\pi}{2}=\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdots,$$

where the numerators are the prime numbers larger than two and the denominators are even numbers not divisible by four and differing by one from the numerators. This result is due to Euler [**24**].

The following three results are special cases of a general algorithm discovered by Ramanujan [**68**]:

$$\frac{\pi}{2}=1+\frac{1}{2}\left(\frac{1}{3}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{5}\right)+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{7}\right)+\cdots, \tag{8-1}$$

$$\frac{\pi}{2}\log 2=1+\frac{1}{2}\left(\frac{1}{3^2}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{5^2}\right)+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{7^2}\right)+\cdots, \tag{8-2}$$

$$\frac{\pi^3}{48}+\frac{\pi}{4}(\log 2)^2=1+\frac{1}{2}\left(\frac{1}{3^3}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{5^3}\right)+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{7^3}\right)+\cdots. \tag{8-3}$$

The first two of these equations were known before Ramanujan.

Another series

$$\frac{4}{\pi}=1+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{1\cdot3}{2\cdot4}\right)^2+\frac{1}{4}\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2+\cdots,$$

is due to E. Catalan [**18**].

A series where the terms are the squares of the arithmetic means of the partial sums of the harmonic series was discovered by H. F. Sandham [**11**]:

$$\frac{17\pi^4}{360}=1+\left(\frac{1+\frac{1}{2}}{2}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\right)^2+\cdots.$$

A similar series is due to Ramanujan [**69**],

$$\frac{\pi^2}{12}=\left(1+\frac{1}{2}\right)\frac{S_2}{2\cdot3}+\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{S_4}{4\cdot5}$$
$$+\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\frac{S_6}{6\cdot7}+\cdots,$$

where S_{2n} is given by (4-17).

The following two series are due to Newton,

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3\cdot2^3}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{5\cdot2^5}\right)+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{7\cdot2^7}\right)+\cdots, \tag{8-4}$$

$$\pi=\frac{3\sqrt{3}}{4}+24\left(\frac{1}{12}-\frac{1}{5\cdot2^5}-\frac{1}{28\cdot2^7}-\frac{1}{72\cdot2^9}-\cdots\right). \tag{8-5}$$

Newton calculated sixteen decimal places of π, of which the last two were incorrect, in 1665–1666 using series (8-5). Most authors, even the scholar F. J. Duarte [**19**], state that Newton did this calculation using (8-4).

Gregory's series (6-1) with $\tan^{-1}x=\pi/6$ gives

$$\pi=2\sqrt{3}\left(1-\frac{1}{3\cdot3}+\frac{1}{5\cdot3^2}-\frac{1}{7\cdot3^3}+\cdots\right). \tag{8-6}$$

Abraham Sharp calculated 72 decimal places of π in 1669 by use of this series. In 1717 the French mathematician F. De Lagny used this series to calculate 126 decimal places of π. He made a mistake in the 113th place obtaining a 7 instead of an 8. This error gave rise to a somewhat comical incident we will see in Section 11.

Ramanujan obtained the series [**70**]

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 - \cdots.$$

The following three series are due to the author [**17**]:

$$\pi = \sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}2^{2n+3}}{(2n+1)(3+\sqrt{5})^{2n+1}}; \tag{8-7}$$

$$\pi = 20\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}^2}{(2n+1)(3+\sqrt{10})^{2n+1}}; \tag{8-8}$$

$$\pi = 12\sqrt{5}\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}}{2n+1}\left[\frac{2(2-\sqrt{3})}{\sqrt{5}+\sqrt{1+16(2-\sqrt{3})}}\right]^{2n+1} \tag{8-9}$$

Ten terms of (8-9) give fifteen decimal figures of π.

Ramanujan obtained the remarkable series [**67**]

$$\frac{1}{\pi} = 2\sqrt{2}\left[\frac{1103}{99^2} + \left(\frac{27{,}493}{99^6}\right)\left(\frac{1}{2}\right)\left(\frac{1\cdot 3}{4^2}\right) + \left(\frac{53{,}883}{99^{10}}\right)\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}\right) + \cdots\right]. \tag{8-10}$$

The numerators of the first fractions of each term above are in arithmetical progression. This series converges extremely rapidly. Three terms of (8-10) give seventeen decimal figures of π! I know of no other series which converges to π faster than (8-10). Surprisingly, though, no one has, to my knowledge, used this series for a high precision calculation of π, even before the advent of Eugene Salamin's algorithm, to be discussed in Section 10.*

The first term of (8-10) gives an interesting approximation to π:

$$\frac{99^2}{2206\sqrt{2}} = 3.141592731\ldots. \tag{8-11}$$

There also exist representations of π by means of infinite continued fractions. For instance, the expression

$$S = a_0 + a_1 + a_1a_2 + a_1a_2a_2 + a_1a_2a_3a_4 + \cdots \tag{8-12}$$

is easily seen to be equivalent to the infinite continued fraction

*(Added in proof.) Since the writing of this paper the author has learned that William Gosper of Symbolics, Inc., of Palo Alto, California, in his evaluation of 17.5 million terms of a continued fraction for π, done in 1985, based his calculation on a careful evaluation of Ramanujan's series (8-10).[1] He used a Symbolics 3670 and his result verified Kanada's evaluation of π to 2^{24} decimal places.

Incidentally, these calculations have discovered that the sequence 314159 appears in the tail of π beginning at the digit 9,973,760.

$$S = a_0 + \frac{a_1}{1} - \frac{a_2}{(1+a_2)} - \frac{a_3}{(1+a_3)} - \cdots. \tag{8-13}$$

Gregory's series (6-1) can, therefore, be written as an infinite continued fraction, by setting $a_0 = 0$, $a_1 = x$, $a_2 = -x^2/3$, $a_3 = -3x^2/5$, etc. We obtain

$$\tan^{-1}x = \frac{x}{1} + \frac{x^2}{(3-x^2)} + \frac{9x^2}{(5-3x^2)} + \frac{25x^2}{(7-5x^2)} + \cdots. \tag{8-14}$$

Setting $x = 1$, we have the result

$$\frac{4}{\pi} = 1 + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \frac{9^2}{2} + \cdots, \tag{8-15}$$

of William, Viscount Brouncker (ca. 1620–1684), the first president of The Royal Society.

J. H. Lambert gave the following infinite continued fraction for π,

$$\pi = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \cdots. \tag{8-16}$$

The first convergents of this continued fraction are [8]:

3	value implied by I Kings vii, 23;
22/7	upper bound given by Archimedes in the 3rd century B.C.;
333/106	lower bound found by Adriaan Anthoniszoon, c. 1583; and
355/113	found by Tsu Chung-Chi, Valentin Otho, and Adriaan Anthoniszoon.

It can be shown [37] that if p/q is a convergent in the expansion of a real number x as a continued fraction, then there does not exist any rational number a/b with $b \leqslant q$ which approximates x better than p/q. This result explains why 22/7 and 355/113 are such popular approximations to π. No fraction with a denominator less than 113 gives a better approximation to π than 355/113, nor is there a fraction with one digit in the denominator, or two digits for that matter, which approximates π better than 22/7. In fact 355/113 is such a good approximation that a better one is not reached until 52,163/16,604!

Monte Zerger [93] noticed the curious fact that the 7th, the 22nd, the 113th, and the 355th digits of π are all 2's. It is also interesting that the decimal expansion of π^2 does not contain a 2 until the 46th decimal place [58]. Notice also the number of 2's in the approximation (5-7).

It was J. H. Lambert, incidentally, who, in 1767, first proved the irrationality of π. Legendre in 1794 proved that π^2 is also irrational. F. Lindemann, in 1882, proved that π is transcendental. Many other proofs of the irrationality and transcendence of π have since appeared, associated with the names of Hermite [32], B. H. Arnold and Howard Eves [5], Hilbert [33], Hurwitz [34], Gordan [28], and E. Landau [46]. A very simple and ingenious proof of the irrationality of π was found by Ivan Niven [56] as recently as 1946.

Charles Hermite proved that e is also transcendental. Then e^{π} was shown by Gelfond to be irrational in 1929. Hilbert, in 1900, propounded the problem of showing that α^{β} is transcendental if α and β are algebraic, α is not 0 or 1, and β is irrational. This problem was solved independently by Gelfond and Schneider in 1934. It shows, in particular, that e^{π}, which is one of the values of i^{-2i}, is transcendental.

It is known that $e\pi$ is irrational. The irrationality of such numbers as π^e, 2^{π}, or $\pi + e$ is still unproved.

Euler gave the following expressions [25]:

$$\pi = 3 + \frac{1^2}{6+}\frac{3^2}{6+}\frac{5^2}{6+}\cdots, \qquad \frac{16}{\pi} = 5 + \frac{1^2}{10+}\frac{3^2}{10+}\frac{5^2}{10+}\cdots,$$

$$\frac{\pi}{2} = 1 + \frac{2}{3+}\frac{1\cdot 3}{4+}\frac{3\cdot 5}{4+}\frac{5\cdot 7}{4+}\cdots, \qquad \frac{\pi}{2} = 1 + \frac{1}{1+}\frac{1\cdot 2}{1+}\frac{2\cdot 3}{1+}\frac{3\cdot 4}{1+}\cdots$$

Ramanujan gave the beautiful expression [71]:

$$\sqrt{\frac{1}{2}\pi e} = 1 + \frac{1}{1\cdot 3} + \frac{1}{1\cdot 3\cdot 5} + \frac{1}{1\cdot 3\cdot 5\cdot 7} + \cdots + \frac{1}{1+}\frac{1}{1+}\frac{2}{1+}\frac{3}{1+}\frac{4}{1+}\cdots.$$

There exist many integral expressions that yield π:

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}dx = \frac{\pi^3}{8},$$

$$\int_0^\infty \frac{dx}{(x^2+11^2)(x^2+21^2)(x^2+31^2)(x^2+41^2)(x^2+51^2)} = \frac{5\pi}{12\cdot 13\cdot 16\cdot 17\cdot 18\cdot 22\cdot 23\cdot 24\cdot 31\cdot 32\cdot 41},$$

$$\int_0^\infty \frac{dx}{(1+x^2)(1+0.001x^2)(1+0.00001x^2)\cdots} = \frac{\pi}{2.2020020002000020000002\cdots},$$

$$\int_0^\infty \frac{\sin(2m+1)\pi x\,dx}{x\{1-(x^2/1^2)\}\{1-(x^2/2^2)\}\cdots\{1-(x^2/n^2)\}} = \frac{\pi}{2}\frac{2^{2n}(n!)^2}{(2n)!},$$

$$\int_0^\infty \frac{\sin(2n+1)x}{\cosh x + \cos x}\frac{dx}{x} = \frac{\pi}{4},$$

$$\int_0^\infty \frac{\sin nx\,dx}{x + \frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots} = \frac{\sqrt{\frac{1}{2}\pi}}{n + \frac{1}{n+}\frac{2}{n+}\frac{3}{n+}\cdots}.$$

The last five integrals are due to Ramanujan [72], [73], [74], [75]. Further,

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi,$$

$$\int_0^\infty \frac{\tan x}{x}dx = \pi,$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2},$$

$$\int_0^\infty \sin^{2n}\theta\,d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi.$$

The formula,

$$e^{ix} = \cos x + i\sin x, \tag{8-17}$$

was first given by Euler in *Miscellanea Berolinensia*, v. 7, 1743, p. 179, (paper read 6

September 1742), and again in his *Introductio in Analysin Infinitorum*, Lausanne, 1748, v. 1, p. 104 [25].

The equivalent of the form

$$ix = \log(\cos x + i \sin x)$$

was given earlier by Roger Cotes in *Philosophical Transactions*, 1714, v. 29, 1717, p. 32.

Ramanujan obtained (8-17) while in the fifth form (high school) and was very disappointed to learn that this result was already known. He kept the paper containing his results secreted in the roofing of his house [79].

If we let $x = \pi$ in (8-17), we get the rather mystifying result,

$$e^{i\pi} + 1 = 0. \tag{8-18}$$

Much melodrama has been written about this equation. Herbert Westren Turnbull [55] attributes to Felix Klein the remark that *all analysis was centered here.* He goes on to add, *Every symbol has its history—the principal whole numbers* 0 *and* 1; *the chief mathematical relations* + *and* = ; π *the discovery of Hippocrates*; *i the sign for the 'impossible' square root of minus one*; *and e the base of Napierian logarithms.*

When the circular and exponential functions are introduced in analysis independent of any geometric constructs, the exponential function of x is defined as the unique solution of the equation,

$$x = \int_1^y \frac{dt}{t}.$$

The base e is defined by

$$1 = \int_1^e \frac{dt}{t}.$$

Similarly, the number π is defined by the equation,

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}.$$

The base e and π are introduced in order to normalize the exponential and trigonometric functions, and (8-18) merely expresses the connection between these bases that a correct definition of the complex exponential function must lead to [2]. I have quoted verbatim from this last reference.

Section 9. Arctangent Formulas for the Calculation of π

Consider the angle α whose tangent is $1/5$:

$$\tan\alpha = \frac{1}{5}.$$

We have then

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{5}{12},$$

and

$$\tan 4\alpha = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{5/6}{1-25/144} = \frac{120}{119}.$$

This value is just 1/119 higher than 1, whose arctangent is $\pi/4$. Since $4\alpha > \pi/4$, we can write the difference of the angles as

$$\beta = 4\alpha - \frac{\pi}{4},$$

and we have

$$\tan\beta = \tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{\tan 4\alpha - \tan(\pi/4)}{1+\tan 4\alpha \tan(\pi/4)} = \frac{\frac{1}{119}}{1+\frac{120}{119}} = \frac{1}{239},$$

which gives, finally,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}, \tag{9-1}$$

a formula discovered by John Machin (1680–1752) in 1706.

If this identity is used together with Gregory's series (6-1), we have

$$\begin{aligned}\pi = &\frac{16}{5}\left(1 - \frac{4}{3\cdot 100} + \frac{4^2}{5\cdot 100^2} - \frac{4^3}{7\cdot 100^3} + \cdots\right)\\ &- \frac{4}{239}\left(1 - \frac{1}{3\cdot 57{,}121} + \frac{1}{5\cdot 57{,}121^2} - \cdots\right),\end{aligned} \tag{9-2}$$

a formula well suited for decimal calculations. With this series Machin calculated 100 decimal figures of π in 1706.

The Greek letter π, to designate the ratio of the circumference to the diameter, was used for the first time in *Synopsis Palmariarum Matheseos* (1706) of William Jones when he published Machin's value of one hundred digits. Before him, William Oughtred (1574–1660) in 1647, and Isaac Barrow (1630–1678) some years later (1669), used the same symbol to designate twice this number. P. Cousin's *Leçons de Calcul Différentiel et de Calcul Intégral*, Paris, 1777, also uses π for the circumference of the circle of unit radius.

Jean Bernoulli (1667–1748) used the letter c to designate π. Euler in 1734 used the letter p. Christian Goldbach (1690–1764) used the letter π in 1742, but it was not until the publication of Euler's famous *Introductio in Analysin Infinitorum*, Lausanne, 1748, that the symbol π was definitely accepted. No doubt π was chosen for being the first letter of the Greek words *perimeter* and *periphery*.

Use of the formulas

$$\tan^{-1}\frac{1}{a-b} = \tan^{-1}\frac{1}{a} + \tan^{-1}\frac{b}{a^2-ab+1} \tag{9-3}$$

and

$$\tan^{-1}\frac{1}{a} = 2\tan^{-1}\frac{1}{2a} - \tan^{-1}\frac{1}{4a^3+3a}, \tag{9-4}$$

together with appropriate choices for a and b allows us either to transform existing identities, or to find new combinations of arcs to obtain π. These identities are simple rewritings of the formula

$$\tan^{-1}z_1+\tan^{-1}z_2=\tan^{-1}\left(\frac{z_1+z_2}{1-z_1z_2}\right).$$

T. J. I'a. Bromwich attributes (9-3) to C. L. Dodgson (Lewis Carroll) [**16**].

The values $a=2$, $b=1$ in (9-3) give the identity, due to Euler,

$$\frac{\pi}{4}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}. \tag{9-5}$$

Together with Euler's series (6-4), this identity gives

$$\begin{aligned}\frac{\pi}{4}&=\frac{4}{10}\left\{1+\frac{2}{3}\left(\frac{2}{10}\right)+\frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{10}\right)^2+\cdots\right\}\\&+\frac{3}{10}\left\{1+\frac{2}{3}\left(\frac{1}{10}\right)+\frac{2\cdot 4}{3\cdot 5}\left(\frac{1}{10}\right)^2+\cdots\right\};\end{aligned} \tag{9-6}$$

$a=3$, $b=1$ in (9-3) give

$$\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}; \tag{9-7}$$

and (9-5) and (9-7) give

$$\frac{\pi}{4}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}, \tag{9-8}$$

published by C. Hutton in *Philosophical Transactions of the Royal Society* in 1776, and by Thomas Clausen in *Astronomischen Nachrichten* in 1847.

Formula (9-8) together with Euler's series (6-4) gives Hutton's series,

$$\begin{aligned}\frac{\pi}{4}&=\frac{6}{10}\left\{1+\frac{2}{3}\left(\frac{1}{10}\right)+\frac{2\cdot 4}{3\cdot 5}\left(\frac{1}{10}\right)^2+\cdots\right\}\\&+\frac{14}{100}\left\{1+\frac{2}{3}\left(\frac{2}{100}\right)+\frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^2+\cdots\right\};\end{aligned} \tag{9-9}$$

$a=7$, $b=4$ in (9-3) gives

$$\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{2}{11}; \tag{9-10}$$

$a=7$, $b=3/2$ in (9-3) gives

$$\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{3}{79}; \tag{9-11}$$

(9-10) and (9-11) together give

$$\tan^{-1}\frac{1}{3}=2\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{3}{79}$$

which, combined with (9-8), gives

$$\frac{\pi}{4}=5\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{3}{79}, \tag{9-12}$$

due to Euler.

Formula (9-12), together with Euler's series (6-4), leads to the highly convergent series,

$$\frac{\pi}{4}=\frac{7}{10}\left\{1+\frac{2}{3}\left(\frac{2}{100}\right)+\frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^{2}+\cdots\right\}$$
$$+\frac{7584}{10^{5}}\left\{1+\frac{2}{3}\left(\frac{144}{10^{5}}\right)+\frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{10^{5}}\right)^{2}+\cdots\right\}. \tag{9-13}$$

With this series Euler calculated 20 decimal places of π in one hour!

It is a striking example of Euler's craft that he was able to obtain a combination of arcs that improves on Machin's identity (9-1) and that, together with *his* series (6-4), reduces the calculation of π to the successive computation of powers of 2 and 12.

In 1877 the Chinese mathematician Tseng-Chi-Hung found 100 decimals of π using Euler's identity (9-5) together with Gregory's series [12].

Not all combinations of arcs that yield π are, of course, equally adequate for a numerical calculation of π. D. H. Lehmer [48] gave a figure of merit for the amount of labor involved in calculating π by the use of arctangent formulas together with *Gregory's series* (6-1). For a relation of the type

$$\frac{k\pi}{4}=\sum_{n=1}^{N} a_{n}\tan^{-1}(1/m_{n}) \tag{9-14}$$

the amount of labor involved is proportional to,

$$\sum_{n=1}^{N}\frac{1}{\log m_{n}}, \tag{9-15}$$

a quantity that Lehmer calls the *measure* of the relation (9-14).

If one of the m_n's is equal to 10, then this term, whose measure is 1, is much easier to calculate than another arctangent of nearly the same size. For this reason Lehmer assigned to 10 the measure $\frac{1}{2}$. If the relation includes, besides 10, the arctangents of other powers of 10, then these can be computed by merely recopying the significant figures in the terms of the series for the arctangent of 10. These arctangents of powers of 10 were assigned, by Lehmer, the measure 0.

The measure of Machin's formula (9-1) is $1/\log 5 + 1/\log 239 = 1.8511$. The measure of (9-5) is 5.4178, and that of (9-8) 3.2792. The measure of Euler's identity (9-12) is 1.8873. Notice that the measure of Euler's identity (9-12) is higher than that of Machin's identity (9-1). It is to be emphasized that these measures refer to the use of Gregory's series (6-1). A glance at (9-13) shows that it is just as simple to evaluate π by use of this series as it is by use of (9-2), and (9-13) converges substantially faster.

One of the methods commonly employed to derive an arctangent relation from another is to replace one arctangent by the sum of two arctangents of larger numbers. Lehmer proved that, in general, this only makes matters worse since this device produces a relation of still higher measure. More precisely, Lehmer proved that if x, y, and z are positive numbers such that

$$\tan^{-1}x=\tan^{-1}y+\tan^{-1}z, \tag{9-16}$$

then

$$\frac{1}{\log x}<\frac{1}{\log y}+\frac{1}{\log z},\quad \text{if}\quad x>2.88200803. \tag{9-17}$$

While condemning in general the substitution of the sum of two arctangents for a single arctangent, Lehmer, nevertheless, points out that in some special cases it may be very desirable. In fact, if one of the new arctangents, say $\tan^{-1}y$ appears elsewhere in the relation, then the measure is decreased by the positive amount

$$\frac{1}{\log x}-\frac{1}{\log z}$$

by this elimination of $\tan^{-1}x$. For example, in general, the measure of a relation containing $\tan^{-1}1/70$ would be increased by 0.3796 if this arctangent were replaced by $\tan^{-1}1/99+\tan^{-1}1/239$. However, the relation

$$\frac{\pi}{4}=12\tan^{-1}\frac{1}{18}+3\tan^{-1}\frac{1}{70}+5\tan^{-1}\frac{1}{99}+8\tan^{-1}\frac{1}{307}, \tag{9-18}$$

due to Bennet, has a measure of 2.2418, and contains both $\tan^{-1}1/70$ and $\tan^{-1}1/99$. Therefore, when we eliminate $\tan^{-1}1/70$ to obtain

$$\frac{\pi}{4}=12\tan^{-1}\frac{1}{18}+8\tan^{-1}\frac{1}{99}+3\tan^{-1}\frac{1}{239}+8\tan^{-1}\frac{1}{307}, \tag{9-19}$$

the measure decreases to 2.1203.

Other considerations might enter into the picture. For example, William Rutherford found it easier to work with the formula,

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}, \tag{9-20}$$

whose measure is 2.4737, rather than with Machin's identity (9-1) because of the ease with which the odd powers of 1/70 and 1/99 can be obtained by hand. These considerations are, of course, unimportant if a computer is used.

In 1824 William Rutherford found 208 digits of π of which the last 56 were incorrect [88]. In 1853 he returned to the calculation, and using (9-20) calculated 440 correct digits of π.

In the following list are given some arctangent formulas for the calculation of π arranged according to the size of the largest arctangent. Each relation is accompanied by its measure if necessary.

L. K. Schulz von Strassnitzky published the relation,

$$\frac{\pi}{4}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8} \quad (5.8599), \tag{9-21}$$

in the *Journal de Crelle* in 1844.

The German calculating prodigy Johann Martin Zacharias Dase, learned from Schulz von Strassnitzky the use of the formula (9-21), and at the age of 20 calculated 205 decimals of π (of which all but the last five were correct), in slightly under two months [38]. Other formulas are:

$$\frac{\pi}{4}=2\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{7} \quad (4.5052), \tag{9-22}$$

due to Euler;

$$\frac{\pi}{4}=3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{5}{99} \quad (2.4322); \tag{9-23}$$

$$\frac{\pi}{4}=3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{20}+\tan^{-1}\frac{1}{1985} \quad (2.7328), \tag{9-24}$$

due to R. W. Morris [9].

D. F. Ferguson, in 1946, calculated 730 decimal places of π with (9-24). Further,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{100} + \tan^{-1}\frac{1}{5000} - \tan^{-1}\frac{1}{10101} \quad (2.7224), \tag{9-25}$$

due to Carl Störmer, published in *L'Intermédiaire des Mathématiciens*, v. II, p. 427. Other formulas are:

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{6} - \tan^{-1}\frac{16}{503} - \tan^{-1}\frac{1}{117} \quad (2.4364); \tag{9-26}$$

$$\frac{\pi}{4} = 6\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{5}{99} - 3\tan^{-1}\frac{1}{268} \quad (2.2904); \tag{9-27}$$

$$\frac{\pi}{4} = 6\tan^{-1}\frac{1}{8} + 2\tan^{-1}\frac{1}{57} + \tan^{-1}\frac{1}{239} \quad (2.0973), \tag{9-28}$$

published by Carl Störmer in 1896;

$$\frac{\pi}{4} = 8\tan^{-1}\frac{1}{10} - \tan^{-1}\frac{1}{239} - 4\tan^{-1}\frac{1}{515} \quad (1.2892), \tag{9-29}$$

attributed to Klingenstierna;

$$\frac{\pi}{4} = 8\tan^{-1}\frac{1}{10} + 3\tan^{-1}\frac{1}{18} + 2\tan^{-1}\frac{1}{100} + 2\tan^{-1}\frac{1}{307} - 3\tan^{-1}\frac{1}{515} + 2\tan^{-1}\frac{1}{9901} \quad (2.3177); \tag{9-30}$$

$$\frac{\pi}{4} = 8\tan^{-1}\frac{1}{10} - 2\tan^{-1}\frac{2543}{452761} - \tan^{-1}\frac{1}{1393}, \quad (1.2624); \tag{9-31}$$

$$\frac{\pi}{4} = 8\tan^{-1}\frac{1}{10} - \tan^{-1}\frac{1}{100} - \tan^{-1}\frac{1}{515} - \tan^{-1}\frac{3583}{371498882} \quad (1.0681); \tag{9-32}$$

$$\frac{\pi}{4} = 7\tan^{-1}\frac{1}{10} + 2\tan^{-1}\frac{1}{50} + 4\tan^{-1}\frac{1}{100} + \tan^{-1}\frac{1}{682} + 4\tan^{-1}\frac{1}{1000} + 3\tan^{-1}\frac{1}{1303} - 4\tan^{-1}\frac{1}{90109} \quad (1.9644), \tag{9-33}$$

due to John W. Wrench, Jr.;

$$\frac{\pi}{4} = 7\tan^{-1}\frac{1}{10} + 8\tan^{-1}\frac{1}{100} + \tan^{-1}\frac{1}{682} + 4\tan^{-1}\frac{1}{1000} + 3\tan^{-1}\frac{1}{1303} - 4\tan^{-1}\frac{1}{90109} - 2\tan^{-1}\frac{1}{500150} \quad (1.5513), \tag{9-34}$$

due also to John W. Wrench, Jr.,

$$\frac{\pi}{4} = 8\tan^{-1}\frac{10}{101} - \tan^{-1}\frac{1}{239} + 4\tan^{-1}\frac{1}{52525} \quad (1.6280); \tag{9-35}$$

$$\frac{\pi}{4} = 12\tan^{-1}\frac{1}{15} - \tan^{-1}\frac{1}{239} - 4\tan^{-1}\frac{10}{4331} \quad (1.6500); \tag{9-36}$$

$$\frac{\pi}{4} = 12\tan^{-1}\frac{1}{18} + 8\tan^{-1}\frac{1}{57} - 5\tan^{-1}\frac{1}{239} \quad (1.7866) \tag{9-37}$$

published by Gauss, *Werke*, v. II, p. 501;

$$\frac{\pi}{4} = 16\tan^{-1}\frac{20}{401} - \tan^{-1}\frac{1}{239} - 4\tan^{-1}\frac{1}{515} + 8\tan^{-1}\frac{1}{1620050} \quad (1.7182); \tag{9-38}$$

$$\frac{\pi}{4} = 22\tan^{-1}\frac{1}{26} - 2\tan^{-1}\frac{1}{2057} - 5\tan^{-1}\frac{38479}{3240647} \quad (1.5279); \tag{9-39}$$

$$\frac{\pi}{4} = 22\tan^{-1}\frac{1}{28} + 2\tan^{-1}\frac{1}{443} - 5\tan^{-1}\frac{1}{1393} - 10\tan^{-1}\frac{1}{11018} \quad (1.6343), \tag{9-40}$$

attributed to E. B. Escott in *L'Intermédiaire des Mathématiciens*, v. III, p. 276;

$$\frac{\pi}{4} = 12\tan^{-1}\frac{1}{38} + 20\tan^{-1}\frac{1}{57} + 7\tan^{-1}\frac{1}{239} + 24\tan^{-1}\frac{1}{268} \quad (2.0348), \tag{9-41}$$

due to Gauss;

$$\frac{\pi}{4} = 44\tan^{-1}\frac{1}{57} + 7\tan^{-1}\frac{1}{239} - 12\tan^{-1}\frac{1}{682} + 24\tan^{-1}\frac{1}{12943} \quad (1.5860), \tag{9-42}$$

due to John W. Wrench, Jr.;

$$\frac{\pi}{4} = 64\tan^{-1}\frac{1}{80} - \tan^{-1}\frac{1}{239} - 4\tan^{-1}\frac{1}{515} - 8\tan^{-1}\frac{1}{4030} - 16\tan^{-1}\frac{1}{32060} - 32\tan^{-1}\frac{1}{256{,}120} \quad (1.9989), \tag{9-43}$$

due to M. Cashmore, *L'Intermédiaire des Mathématiciens*, v. XXVI, p. 22;

$$\frac{\pi}{4} = 78\tan^{-1}\frac{1}{100} - 2\tan^{-1}\frac{1}{682} + 3\tan^{-1}\frac{1}{5396} + 10\tan^{-1}\frac{243}{345737} - 17\tan^{-1}\frac{1}{62575} - 34\tan^{-1}\frac{1}{500150} \quad (1.6112), \tag{9-44}$$

$$\frac{\pi}{4} = 160\tan^{-1}\frac{1}{200} - \tan^{-1}\frac{1}{239} - 4\tan^{-1}\frac{1}{515} - 8\tan^{-1}\frac{1}{4030} - 16\tan^{-1}\frac{1}{50105} - 16\tan^{-1}\frac{1}{62575} - 32\tan^{-1}\frac{1}{500150} - 80\tan^{-1}\frac{1}{4000300} \quad (2.2494), \tag{9-45}$$

due to Bennet;

$$\frac{\pi}{4} = 3\tan^{-1}\frac{1}{239} + 236\tan^{-1}\frac{1}{307} - 12\tan^{-1}\frac{1}{19703} + 240\tan^{-1}\frac{1}{93943} + 8\tan^{-1}\frac{3}{10099} - 24\tan^{-1}\frac{272}{36101879} - 80\tan^{-1}\frac{816}{2922754103} \quad (1.8878); \tag{9-46}$$

$$\frac{\pi}{4} = 2805\tan^{-1}\frac{1}{5257} - 398\tan^{-1}\frac{1}{9466} + 1950\tan^{-1}\frac{1}{12943} + 1850\tan^{-1}\frac{1}{34208} + 2021\tan^{-1}\frac{1}{44179} + 2097\tan^{-1}\frac{1}{85353} + 1484\tan^{-1}\frac{1}{114669} + 1389\tan^{-1}\frac{1}{330182} + 808\tan^{-1}\frac{1}{485298} \quad (1.9568), \tag{9-47}$$

due to Gauss.

Use of the preceding identities provides rapid ways of determining the value of π. The more rapidly the arctangent series converges, the faster the calculation will be. By way of comparison we did a computer run of Machin's identity (9-1) using Gregory's series (6-1), Euler's series (6-4), and our series (7-3), (7-5), and (7-7). The results were:

n	Gregory	Euler	Series (7-3)
0	3.183263598326360	3.060186968243409	3.158065244130476
1	3.140597029326061	3.139082236428362	3.141398624667959
2	3.141621029325035	3.141509789149037	3.141595487171593
3	3.141591772182177	3.141589818359699	3.141592608238813
4	3.141592682404400	3.141592554401089	3.141592654354197
5	3.141592652615309	3.141592650066872	3.141592653576465
6	3.141592653623555	3.141592653463290	3.141592653590032
7	3.141592653588603	3.141592653585213	3.141592653589789
8	3.141592653589836	3.141592653589626	3.141592653589794
9	3.141592653589792	3.141592653589787	3.141592653589793
10	3.141592653589794	3.141592653589793	
11	3.141592653589793		

n	Series (7-5)	Series (7-7)
0	3.148158616418292	3.144216894790728
1	3.141554182069219	3.141584979408989
2	3.141592944101887	3.141592683659296
3	3.141592651171905	3.141592653459185
4	3.141592653611002	3.141592653590392
5	3.141592653589601	3.141592653589791
6	3.141592653589795	3.141592653589793
7	3.141592653589793	

We see that our series consistently converge faster than either Gregory's or Euler's series. The root of equation (7-8) corresponding to $\alpha = 1/5$ is 63.25229744727801..., and the one corresponding to $\alpha = 1/239$ is 2987.51589950963.... These roots were used in the calculation above corresponding to equation (7-7).

Georg von Vega (1756–1802) used Hutton's series (9-9) in 1789 to find the value of π with 143 decimals of which 126 were correct. In 1794 he went back to the calculation and found, with the same formula, 140 decimal places of π of which 136 were correct [87].

At the Radcliffe Library in Oxford there exists an anonymous document giving the value of π with 154 decimals of which 152 are correct [53]. Thibaut found 156 decimals of π in 1822 [29] and Thomas Clausen (1801–1885) published 248 decimals of π in 1847 [7].

William Shanks, using Machin's identity, calculated 318 digits of π in 1853, in the same year he raised this value to 530 digits, then to 607, and finally to 707 digits in 1873–74 [63]. In 1945, D. F. Ferguson found an error in Shanks' calculation from the 527th place onward.

In 1853, Professor Richter, of Elbing, gave 330 digits of π, the next year he gave 440, and finally, 500 in 1855 [6].

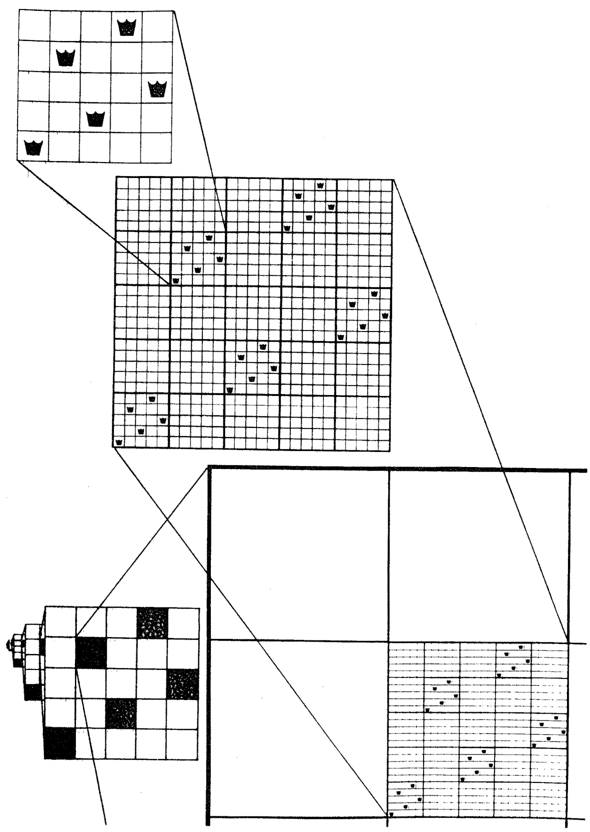
After Ferguson's calculation of 730 decimal figures of π in 1946, the Americans John W. Wrench, Jr. and Levi B. Smith, using Machin's identity, carried the approximation to 808 decimal places [51].

With the advent of the electronic computer the number of known decimals rose rapidly. By 1958, 10,000 decimal places were known. In July 1961, Daniel Shanks and John W. Wrench, Jr., with an IBM 7090 at the IBM Data Processing Center in New

York, using (9-28), found 100,265 decimal places of π, of which the first 100,000 were published [80]. The total time of computation was 8 hours and 43 minutes. By 1973, one million decimal places were known, and, because of the discovery to be discussed in the next section, in 1983 came the heavy artillery.

To be continued in the next issue.

Proof without Words: Inductive Construction of an Infinite Chessboard with Maximal Placement of Nonattacking Queens



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—Dean S. Clark and Oved Shisha
University of Rhode Island

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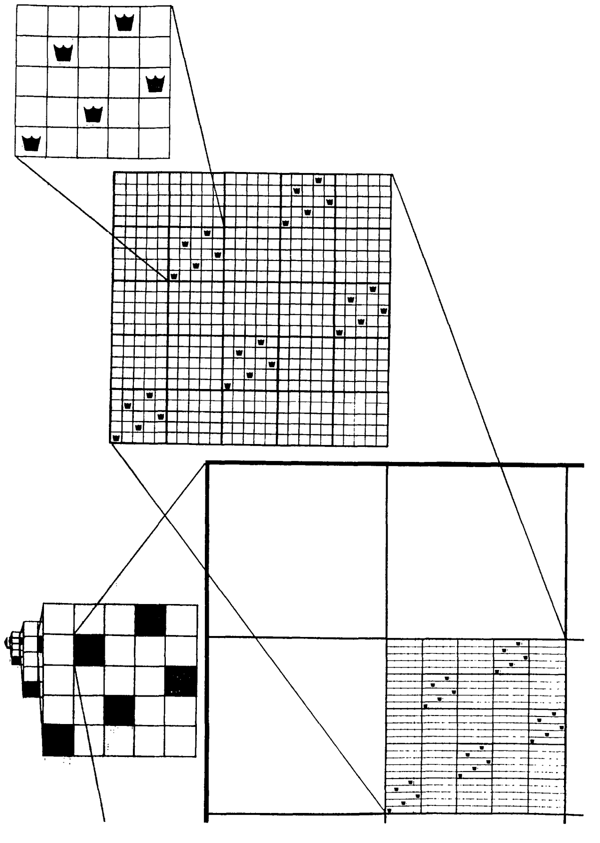

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Who Is the Jordan of Gauss-Jordan?

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Three recent linear algebra texts credit the Jordan half of the Gauss-Jordan method of elimination to the famous French algebraist, Camille Jordan (1838–1922) [1], [5], [7]. This is not surprising, since he is responsible for several other important mathematical concepts, e.g., the Jordan canonical form in linear algebra, the Jordan curve theorem in topology, and the Jordan-Hölder theorem in group theory. But the modification of Gauss's elimination scheme for solving systems of linear equations is due to a different Jordan, the German geodesist, Wilhelm Jordan (1842–1899), who first published the method in his major work, *Handbuch der Vermessungskunde* (*Handbook of Geodesy*) [6]. The purpose of this note is to return the credit to the correct Jordan and to describe the background for his invention.

Methods for solving systems of linear equations had been known for centuries; in fact, the method named for Gauss of finding a triangular system equivalent to the original square one was known to the Chinese some 2000 years ago [8]. On the other hand, Gauss's interest in the subject stemmed from the necessity of solving fairly large systems of equations which arose in connection with his method of least squares. This latter method, which Gauss invented before he was 20 years old, enables one to find the linear function which best fits a set of observed data points. This is equivalent to finding the best approximate solution x to a system $Ax=b$, where A is an $m\times n$ matrix of rank $n, m>n$, and b is a known m-vector. Since such a system is overdetermined, we do not expect it to have an exact solution. By the "best approximate solution" we mean one that minimizes the length $|Ax-b|$ of the error vector. It turns out that this solution x is given by the exact solution to the $n\times n$ system $(A^tA)x=A^tb$. (See [2] for a simple treatment of the method of least squares.) It was linear systems of this form that Gauss needed to solve (though he did not use matrix terminology).

Gauss was interested in the method of least squares particularly for finding approximate solutions to problems in astronomy and geodesy. In fact, he presented his method of elimination in a paper dealing with the asteroid Pallas in 1811 [3]. He had made numerous observations of the asteroid over a period of 6 years; from these he wanted to compute the six basic parameters of the orbit, e.g., the eccentricity of the ellipse and the inclination of the plane of the orbit. From observation and astronomical theory Gauss determined twelve linear equations connecting the six unknowns. The method of least squares then led him to a 6×6 system of equations which determined the best solution. His elimination method gave him a systematic way of solving these equations.

Wilhelm Jordan, in his work on surveying, also had to use the method of least squares regularly. As in astronomy, when one makes geodetic observations there is some redundancy in angle and length measurements. However, there are always various conditions connecting the given measurements, and these can be written in terms of an overdetermined system of linear equations to which one applies this method. Jordan himself was involved in some large scale surveys in Germany as well as in the first survey of the Libyan desert. In 1873 he founded the German *Journal of Geodesy* and in the same year published the first edition of his famous *Handbook*. (There does not seem to be any English biography of Jordan; there are brief biographical sketches in various German encyclopedias and biographical collections.)

Since the method of least squares was so crucial in surveying, Jordan devoted the first section of his *Handbook* to this topic. As part of the discussion, he gave a detailed presentation of Gauss's method of elimination to convert the given system to a triangular system. He then showed how the technique of "back-substitution" enabled one to find the actual solution when numerical coefficients were given. However, he said, "if one carries out this substitution not numerically, but algebraically," one can get the solutions for the unknowns in terms of formulas involving the original coefficients. In the first and second (1879) editions of his book, he simply gave these formulas, but by the fourth edition (1895) he was able to give an explicit algorithm on how to solve a symmetric system of linear equations. (Of course, since A^tA is always symmetric, it was only symmetric systems which occurred in the method of least squares.) This algorithm is, in effect, the Gauss-Jordan method.

Though Jordan did not use matrices as we do, he did set up the work in the form of tables of coefficients and explain how to get from one line to the next, much as current texts do, as he converted from the original system to the simplest equivalent one. The major difference between his method and the current one is that Jordan does not make the initial coefficient of each row 1 during the solution procedure. In the final step he simply expresses each unknown as a quotient with that initial coefficient as denominator.

Jordan's *Handbook* became a standard work in the field of geodesy, going through ten German editions as well as translations into other languages. Even the eighth edition of 1935 had the mathematical first section with the description of the Gauss–Jordan method. The most recent edition, however, published in 1961, eliminated this section. Of course, by this time much of what Jordan himself had written had been modified beyond recognition by a series of editors.

Before the mid-1950's, most of the references to the Gauss-Jordan method were in books or articles about numerical methods. It is only in recent decades that the method has appeared in elementary linear algebra texts. Most such texts, even when they name the method, make no reference at all to the inventor. The only recent text I have found with the correct attribution unfortunately gives incorrect dates for Jordan's birth and death [4]. I hope that this note will provide some recognition to the true developer of the most commonly taught method of solving linear equations.

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Money is Irrational

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Here is a trick for folding a U.S. dollar bill first into an equilateral triangle and then into a tetrahedron.

Step 1: Fold a dollar in half lengthwise, crease it, and then unfold it and lay it face up on a flat surface (see FIGURE 1).

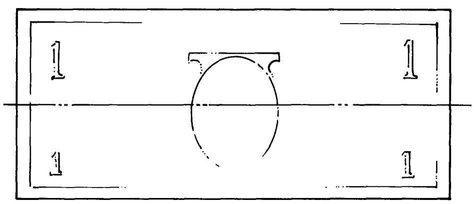

FIGURE 1

Step 2: Fold the upper left-hand corner down until it just touches the crease (see FIGURE 2). It is an elementary exercise in plane geometry to prove that the triangle folded over is a 30°-60°-90° triangle.

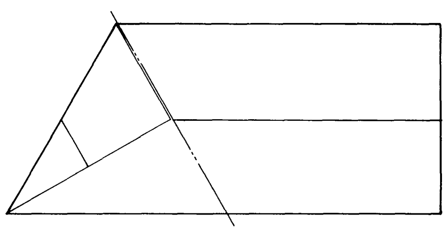

FIGURE 2

Step 3: Fold the lower left-hand corner up until it just touches the top of the bill, as in FIGURE 3.

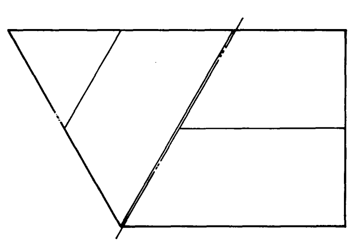

FIGURE 3

Step 4: Fold what is now the upper left-hand corner down to touch the bottom of the bill (see FIGURE 4). Remarkably, it will end up at or very near the lower right-hand corner.

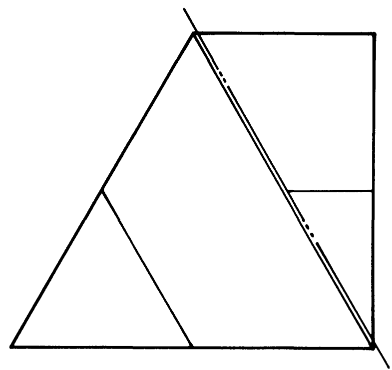

FIGURE 4

Step 5: Fold the remaining tab over, to form an equilateral triangle.

This demonstrates that the ratio of the height of a dollar to its length is approximately the ratio of the altitude of an equilateral triangle to two bases, an irrational ratio of $\sqrt{3}:4$.

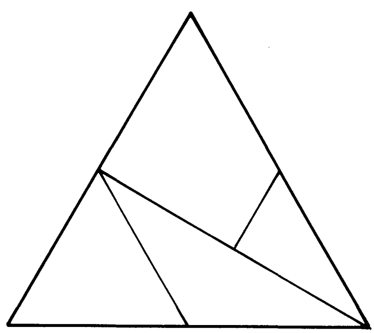

FIGURE 5

If all of the creases have been sharp, you can now gently open up the dollar bill, and when you put it down it will form a tetrahedron (with a picture of a pyramid on one side!)

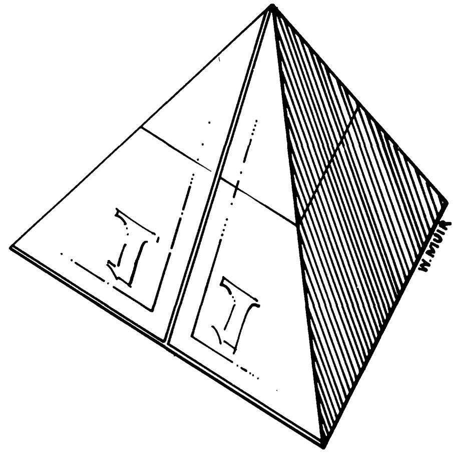

FIGURE 6

The Secretary's Packet Problem

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The secretary's problem is to determine the probability that p letters randomly inserted into p envelopes will all be misaddressed. This problem is solved by using derangements and inclusion-exclusion. Consider the following variation: Each day p letters and envelopes are prepared, and the envelopes are kept in their proper order. The letters are randomly permuted but not inserted in the envelopes. To make matters worse, after n days the n packets of p letters are randomly permuted. When the letters are finally put into the envelopes (which have remained in chronological order), what is the probability that all letters will be misaddressed? We call this the secretary's packet problem. If $p = 1$ or $n = 1$, then the secretary's packet problem reduces to the secretary's problem.

We first recall some basic facts about permutations. Let S_n be the set of all permutations of $\{1,2,3,\ldots,n\}$. A permutation π is a derangement if for all i, $\pi(i) \neq i$ and D_n is the number of derangements of S_n. Then D_n has the following properties [1]

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots \pm \frac{1}{n!}\right). \tag{1}$$

$$D_n - nD_{n-1} = (-1)^n. \tag{2}$$

$$D_n \sim n! \cdot e^{-1}. \tag{3}$$

The answer to the secretary's first problem is $D_p/p!$, which is approximately e^{-1} by formula 3.

Let G be a group and H a subgroup of S_n. The elements of $G \int H$, the wreath product of G and H, are all $(n+1)$-tuples $(e_1, e_2 \ldots, e_n, \pi)$, where $e_i \in G$ and $\pi \in H$. $G \int H$ is actually a group, but we do not need the group structure here. Wreath products are useful for constructing representations of the symmetric group, see [2].

In the packet problem, we take $G = S_p$ and e_i to be the permutation of the p letters written on the ith day. Further, H is S_n, the set of all permutations of days. A member of $S_p \int S_n$ determines some ordering of the pn letters, and so can be considered to be an element of S_{pn}. There are $(p!)^n n!$ permutations in $S_p \int S_n$. Let $T_{p,n}$ be the number of derangements in $S_p \int S_n$.

Theorem.

(a) $T_{p,n} = (p!)^n n! \sum_{k=0}^{n} (D_p/p! - 1)^k/k!$

(b) $T_{p,n} - np!T_{p,n-1} = (D_p - p!)^n$.

(c) $T_{p,n} \sim (p!)^n n! \exp(D_p/p! - 1)$ *for p and n large.*

COROLLARY. *The probability of all the letters misaddressed for the packet problem is approximately*

$$e^{e^{-1}-1} = .53416\ldots.$$

Proof of Theorem. There are $\binom{n}{k} D_{n-k}$ permutations π in S_n with exactly k fixed points. Consider the permutation of $S_p \int S_n$ corresponding to $(e_1, e_2, \ldots, e_n, \pi)$. If $\pi(i) \neq i$, then all letters in the ith packet are misaddressed, regardless of e_i. If $\pi(i) = i$, then the ith packet is in the correct place, and so if all the letters of the ith packet are to be misaddressed, e_i must be a derangement. The number of derangements in $S_p \int S_n$ where π has exactly k fixed points is thus

$$(D_p)^k (p!)^{n-k} \binom{n}{k} D_{n-k}. \tag{4}$$

Summing (4) over all possible k, we get

$$T_{p,n} = \sum_{k=0}^{n} (D_p)^k (p!)^{n-k} \binom{n}{k} D_{n-k} = \sum_{k=0}^{n} (D_p)^{n-k} \binom{n}{k} D_k (p!)^k.$$

To simplify the proof, we introduce the polynomials

$$U_n(x) = \sum_{k=0}^{n} D_k \binom{n}{k} x^k.$$

Since $T_{p,n} = D_p^n U_n(p!/D_p)$, (a) and (b) follow easily from

$$\text{(a}'\text{)}\quad U_n(x) = n! \sum_{k=0}^{n} (k!)^{-1} (1-x)^k x^{n-k}$$

$$\text{(b}'\text{)}\quad U_n(x) - nxU_{n-1}(x) = (1-x)^n.$$

The proof of (b′) uses formula (2)

$$\begin{aligned} U_n(x) - nxU_{n-1}(x) &= \sum_{k=0}^{n} \binom{n}{k} D_k x^k - nx \sum_{k=0}^{n-1} \binom{n-1}{k} D_k x^k \\ &= \sum_{k=0}^{n} x^k \binom{n}{k} [D_k - kD_{k-1}] \\ &= \sum_{k=0}^{n} x^k \binom{n}{k} (-1)^k \\ &= (1-x)^n. \end{aligned}$$

We prove (a′) inductively. Since $D_0 = 1$, the formula holds for $n = 0$. If (a′) is true for $n-1$ then, from (b′),

$$\begin{aligned} U_n(x) &= nxU_{n-1}(x) + (1-x)^n \\ &= nx(n-1)! \sum_{k=0}^{n-1} (k!)^{-1} (1-x)^k x^{n-1-k} + (1-x)^n \\ &= n! \sum_{k=0}^{n} (k!)^{-1} (1-x)^k x^{n-k}. \end{aligned}$$

Then (c) follows easily from (a). For the corollary, the probability is $T_{p,n}/(n!(p!)^n)$, which by (c) is approximately $\exp(D_p/p! - 1)$. As p increases, by (3) this is approximately $\exp(e^{-1} - 1)$.

The following table contains the probability $P_{n,p} = T_{n,p}/n!(p!)^n$ for various values of n and p. We see rapid convergence to $e^{e^{-1}-1} = .53146\ldots$. When $p = 1$, (a) reduces to $T_{1,n} = D_n$, and when $n = 1$, (a) reduces to $T_{p,1} = D_p$. This is evident in the table, where the first row and column are converging to $e^{-1} = .367879\ldots$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$p=1$	0	0.5	0.33333	0.375	0.36666	0.36805	0.36785	0.36788
$p=2$	0.5	0.625	0.60416	0.60677	0.60651	0.60653	0.60653	0.60653
$p=3$	0.33333	0.55555	0.50617	0.5144	0.5133	0.51342	0.51341	0.51341
$p=4$	0.375	0.57031	0.52962	0.53598	0.53518	0.53526	0.53526	0.53526
$p=5$	0.36666	0.56722	0.52488	0.53158	0.53073	0.53082	0.53081	0.53081
$p=6$	0.36805	0.56773	0.52567	0.53231	0.53147	0.53156	0.53155	0.53155
$p=7$	0.36785	0.56765	0.52555	0.53221	0.53137	0.53146	0.53145	0.53145
$p=8$	0.36788	0.56766	0.52557	0.53222	0.53138	0.53147	0.53146	0.53146

Exercise 1. Suppose that instead of counting the number of derangements in $S_p \int S_n$, we ask for the number of derangements in $K \int S_n$, where K is an arbitrary subset of S_p. Let D be the number of derangements in K, and P the number of permutations in K. Prove that the analogs of (a), (b), and (c) are true, where $p!$ is replaced by P and D_p by D. Show that the probability of finding a derangement is approximately $e^{D/P-1}$.

Exercise 2. What happens if the secretary types p letters a day, and puts them randomly into a packet? After n days, he puts them randomly into a drawer. After m drawers are full, he pulls the drawers out in some random fashion, and then starts stuffing envelopes—all of which have stayed in their proper order. Show that the probability that all the letters are misaddressed is $\exp(\exp(\exp(-1) - 1) - 1)$.

Exercise 3. Define $F(1) = e^{-1}$ and $F(k) = \exp(F(k-1) - 1)$ for $k \geqslant 2$. If we have w storage places (packets, drawers, desks, rooms,...), show that the probability that all letters are misaddressed is, for large enough parameters, approximately $F(w)$. Show that $F(w)$ converges to 1 as w goes to infinity.

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The Osculating Spiral

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This article describes a method of approximating nonplanar space curves by cylindrical spirals rather than by planar circles. Readers familiar with osculating circle approximations should skip to the middle of the article. Readers unfamiliar with osculating circles will only need some familiarity with the geometry of vectors in three dimensions, especially the dot and cross products, along with some knowledge of parameterization of curves in space.

One way to approximate a space curve by a more simple curve is to compare it with the so-called osculating circle. [1]

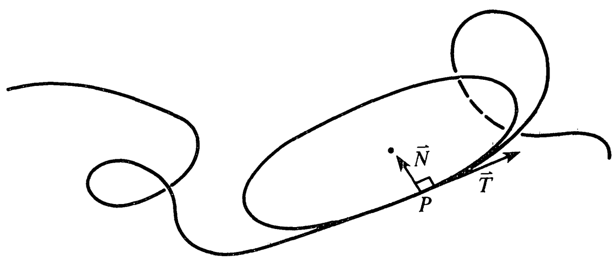


FIGURE 1

In Figure 1 you can see that near any given point, P, on a space curve, the circle which imitates the curve most closely is the one which:

(1) is tangent to the curve at P,

(2) lies in the plane of $\vec{T}$ and $\vec{N}$, the unit tangent and normal vectors to the curve at P, and

(3) has the same curvature (tendency to bend) as the curve does at P.

The curvature of a space curve at any point, P, is: $\kappa = \|d\vec{T}/ds\|$, where s stands for arclength. From this you see that κ measures the rate at which $\vec{T}$ changes direction as you move away from P along the curve. If the curve is a circle of radius a, then as you move along the curve, whenever $\vec{T}$ changes direction by θ radians, the amount of arclength traversed will be $a\theta$ units (as shown in Figure 2). Therefore, the rate at which $\vec{T}$ tends to bend away from its direction at any given moment (i.e., the curvature of a circle of radius a) is:

$$\kappa = \frac{\theta \text{ radians}}{a\theta \text{ units of arclength}} = \frac{1}{a}.$$

Thus, to make a circle with a given curvature you must make the radius equal to the reciprocal of the desired curvature, i.e., the radius of the osculating circle must $= 1/\kappa$.

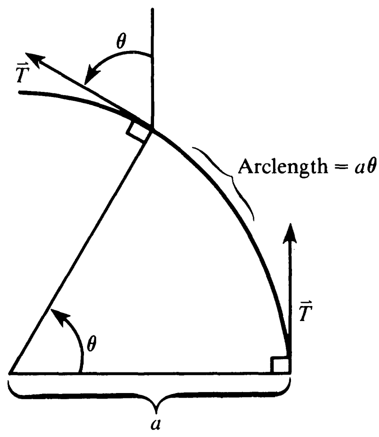


FIGURE 2

The above derivation of the curvature of a circle (and, hence, of the proper radius for an osculating circle) may seem a bit magical if you have not dealt closely with these things before. Let us take the curvature of a circle by the definition, viz, $\kappa = \|d\vec{T}/ds\|$:

Let

$$\vec{R} = x(t)\vec{i} + y(t)\vec{j} = a\cos(t)\vec{i} + a\sin(t)\vec{j}.$$

Then,

$$d\vec{R}/dt = x'(t)\vec{i} + y'(t)\vec{j} = -a\sin(t)\vec{i} + a\cos(t)\vec{j}$$

is a tangent vector to the curve $\left(\text{slope} = \frac{dy}{dt}\Big/\frac{dx}{dt} = dy/dx\right)$ and

$$\|d\vec{R}/dt\| = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(dx)^2 + (dy)^2}\,/dt = ds/dt.$$

So, here,

$$\frac{ds}{dt} = \left\|\frac{dR}{dt}\right\| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2} = a.$$

By construction, the *unit* tangent vector, $\vec{T}$, would be:

$$\frac{d\vec{R}/dt}{\|d\vec{R}/dt\|}.$$

Therefore:

$$\vec{T} = \frac{-a\sin(t)\vec{i} + a\cos(t)\vec{j}}{a} = -\sin(t)\vec{i} + \cos(t)\vec{j}.$$

Now,

$$d\vec{T}/ds = \frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}.$$

So, here:

$$\frac{d\vec{T}}{ds} = \left[-\cos(t)\vec{i} - \sin(t)\vec{j}\right]/a.$$

Finally,

$$\kappa = \|d\vec{T}/ds\| = (1/a)\sqrt{(-\cos(t))^2 + (-\sin(t))^2} = 1/a,$$

a little anticlimactic but satisfying nonetheless.

To help solidify the ideas presented so far, let us calculate an osculating circle for a well-known curve at a convenient point:

$$y = x^2 \text{ at the point } x = y = 0.$$

Let $\vec{R} = t\vec{i} + t^2\vec{j}$ parameterize the curve. Then,

$$\vec{R}' = 1\vec{i} + 2t\vec{j}$$

is a tangent vector and

$$\|\vec{R}'\| = ds/dt = \sqrt{1^2 + (2t)^2}.$$

Therefore,

$$\vec{T} = \vec{R}'/\|\vec{R}'\| = \left[1\vec{i} + 2t\vec{j}\right]/\sqrt{1+4t^2}$$

and

$$d\vec{T}/ds = \frac{d\vec{T}/dt}{ds/dt} = \frac{\left[-4t/(1+4t^2)^{3/2}\right]\vec{i} + \left[2/(1+4t^2)^{3/2}\right]\vec{j}}{\sqrt{1+4t^2}}.$$

So,

$$\kappa = \|d\vec{T}/ds\| = \sqrt{16t^2+4}\,/(1+4t^2)^2 = 2\sqrt{4t^2+1}\,/(1+4t^2)^2 = 2/(1+4t^2)^{3/2}.$$

At $t = 0$, $\kappa = 2/1 = 2$. Therefore, the radius of curvature is $a = 1/2$.

From geometry we know that a tangent to a circle is always perpendicular to the radius at the point of tangency. Since $\vec{T}$ is supposed to be tangent to the osculating circle (which is tangent to the curve) and $\vec{N}$ is defined to be perpendicular to $\vec{T}$, the center of curvature must lie in the direction of $\vec{N}$ as shown in FIGURE 3:

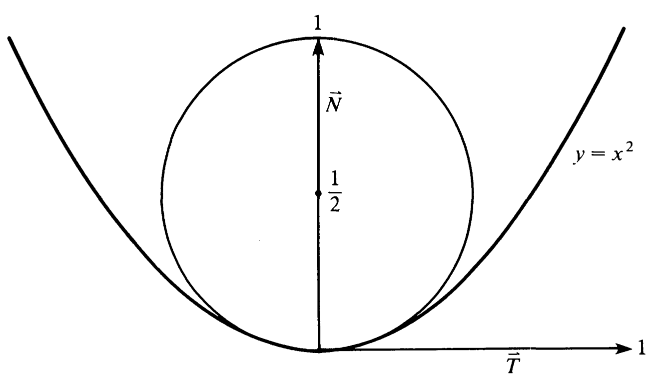


FIGURE 3

In the example just given it was easy to find $\vec{N}$ because $\vec{T}$ was horizontal. Also, in general, in R^2, if $\vec{T} = a\vec{i} + b\vec{j}$, then $\vec{N} = \pm(-b\vec{i} + a\vec{j})$. However, no such simple trick in R^3 can turn $\vec{T}$ into $\vec{N}$. Fortunately, the following argument shows that $d\vec{T}/ds$ is perpendicular to $\vec{T}$ (and thus, in the general case, will provide a formula for obtaining $\vec{N}$):

$\|\vec{T}\|^2 = 1$	$\vec{T}$ is a unit vector
$\vec{T} \cdot \vec{T} = 1$	substitution
$\frac{d}{ds}(\vec{T} \cdot \vec{T}) = \frac{d}{ds}(1) = 0$	derivative of a constant is 0
$d\vec{T}/ds \cdot \vec{T} + \vec{T} \cdot d\vec{T}/ds = 0$	product rule for differentiation applies to the standard dot product in R^2 or R^3
$2(\vec{T} \cdot d\vec{T}/ds) = 0$	dot product is commutative
$\vec{T} \cdot d\vec{T}/ds = 0$	$2 \neq 0$

This last statement and the fact that $\kappa = \|d\vec{T}/ds\|$ show that:

$$\vec{N} = \frac{d\vec{T}/ds}{\|d\vec{T}/ds\|} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

is a *unit* vector perpendicular to $\vec{T}$.

Now you have κ giving a measure of the *rate* at which $\vec{T}$ changes direction as you move along the curve, and $\vec{N} = d\vec{T}/ds$ showing the *direction* towards which $\vec{T}$ is deflecting, i.e., $\vec{N}$ points towards the center of curvature.

Obviously, as shown in FIGURES 1 and 3, the osculating circle is a very good and very simple way to approximate the complexities of a given curve near a given point. However, each nonplanar curve in 3-space has, at each point, a certain torsion (tendency to bend away from the plane of $\vec{T}$ and $\vec{N}$). The osculating circle, being planar, can imitate curvature but not torsion. Copying curvature turns out to be insufficient to capture the spirit of a curve in space. In fact, as we are about to see by considering the cylindrical spiral, it is possible to have infinitely many curves all with exactly the same curvature at every point and yet no two of which parallel one another because they all differ in torsion. (Torsion will be quantified later. For now just think of it as the tendency to be nonplanar.)

Let us consider the cylindrical spiral:

$$\vec{R} = a\cos(t)\vec{i} + a\sin(t)\vec{j} + btk, \qquad a > 0$$

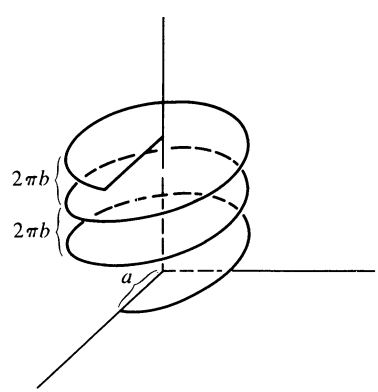


FIGURE 4

$$\vec{T}=\frac{\vec{R}'}{\|\vec{R}'\|}=\frac{-a\sin(t)\vec{i}+a\cos(t)\vec{j}+b\vec{k}}{\sqrt{a^2\sin^2(t)+a^2\cos^2(t)+b^2}}=\frac{-a\sin(t)\vec{i}+a\cos(t)\vec{j}+b\vec{k}}{\sqrt{a^2+b^2}}.$$

Note: $ds/dt=\|\vec{R}'\|=\sqrt{a^2+b^2}$

$$\frac{d\vec{T}}{ds}=\frac{d\vec{T}/dt}{ds/dt}=\frac{\left[-a\cos(t)\vec{i}-a\sin(t)\vec{j}\right]/\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$$

$$\kappa=\|d\vec{T}/ds\|=\frac{1}{a^2+b^2}\sqrt{a^2\cos^2(t)+a^2\sin^2(t)}=\frac{a}{a^2+b^2}$$

$$\vec{N}=\frac{1}{\kappa}\frac{d\vec{T}}{ds}=-\cos(t)\vec{i}-\sin(t)\vec{j}.$$

Now suppose you want a family of curves all with curvature $\kappa=1$. All you have to do is let $b^2=a-a^2$, i.e., $b=\sqrt{a-a^2}$. Then let a range over the interval $(0,1]$ and the curves thus generated:

$$R=(1-a+a\cos(t))\vec{i}+a\sin(t)\vec{j}+\sqrt{a-a^2}\,t\vec{k}$$

will all pass through $(1,0,0)$ when $t=0$ and all will have constant curvature $\kappa=1$ but obviously all these curves will differ from one another (see FIGURE 5).

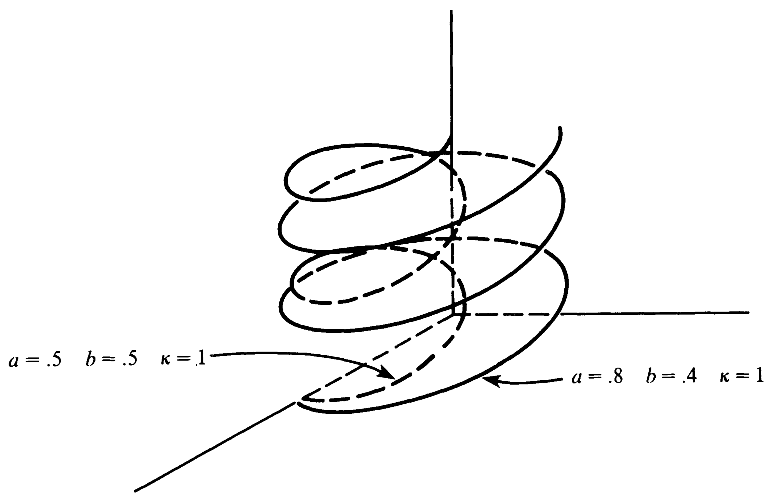


FIGURE 5

There is no way to tilt the one curve shown in FIGURE 5 so as to make it coincide along its whole length with the other curve. The reason for this is that they differ in torsion, though their curvatures do match.

Torsion can be quantified by determining how great a tendency there is for $\vec{N}$ to bend away from the plane of $\vec{T}$ and $\vec{N}$ as you move along the curve. That is, to measure torsion you want to know how much of the vector $d\vec{N}/ds$ *does not* lie in the plane of $\vec{T}$ and $\vec{N}$. Therefore, you want to know how much of $d\vec{N}/ds$ points in the direction of $\vec{T}\times\vec{N}=\vec{B}$ (the unit binormal which points up out of the plane of $\vec{T}$ and $\vec{N}$). Torsion, then, is defined as:

$$\tau=d\vec{N}/ds\cdot\vec{B}$$

A well-established theory of space curves, their curvature and torsion [2] shows that any two curves with the following properties *must be identical*:

(1) They are parameterized by some variable, say, t.

(2) Both pass through point P at the same value of t (say, $t = 0$).

(3) At P, both curves have the same $\vec{T}$ and $\vec{N}$.

(4) For each value of t, curvature and torsion of the two curves agree.

(N. B. Parameterization was introduced here only to insure that a comparison of the two hypothetical curves, point by point, would be easier for the reader to visualize. Curvature and torsion are descriptions of the curve as it lies in space and hence independent of any particular parameterization.)

Inspired by this knowledge of the power of torsion to describe curves more fully, we now calculate τ for a cylindrical spiral and hope that knowing $\vec{T}$, $\vec{N}$, κ, and τ at a point, P, on a given space curve, a way can be found to fit a spiral to the curve so that at P the spiral will have the same $\vec{T}$, $\vec{N}$, κ, and τ as the curve does. If this can be accomplished then, in a sense, there will be no simple curve which imitates the given curve near P better than our "osculating spiral" does because curvature and torsion essentially determine a curve. Here goes.

First:

$$\vec{B} = \vec{T} \times \vec{N} = \left[\frac{-a\sin(t)\vec{i} + a\cos(t)\vec{j} + b\vec{k}}{\sqrt{a^2+b^2}}\right] \times \left[-\cos(t)\vec{i} - \sin(t)\vec{j}\right]$$

$$= \frac{b\sin(t)\vec{i} - b\cos(t)\vec{j} + a\vec{k}}{\sqrt{a^2+b^2}}.$$

Then:

$$d\vec{N}/ds = \frac{d\vec{N}/dt}{ds/dt} = \frac{\sin(t)\vec{i} - \cos(t)\vec{i}}{\sqrt{a^2+b^2}}.$$

So:

$$\tau = d\vec{N}/ds \cdot \vec{B} = \frac{b}{a^2+b^2}.$$

Notice that:

$$\kappa^2 + \tau^2 = \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{1}{a^2+b^2}.$$

Therefore:

$$\frac{\kappa}{\kappa^2+\tau^2} = a \quad \text{and} \quad \frac{\tau}{\kappa^2+\tau^2} = b.$$

Thus, it turns out to be fairly easy to manufacture a spiral with a given curvature and torsion. If you want $\kappa = 1$ and $\tau = 2$, e.g., then in your spiral formula you need:

$$a = \frac{1}{1^2+2^2} = \frac{1}{5} \quad \text{and} \quad b = \frac{2}{1^2+2^2} = \frac{2}{5}.$$

The only problem remaining is to find a way to match $\vec{T}$ and $\vec{N}$ of the spiral to $\vec{T}$ and $\vec{N}$ of the curve to be approximated. To do this, first note that in the standard position of the spiral:

$$\vec{R} = a\cos(t)\vec{i} + a\sin(t)\vec{j} + bt\vec{k} \quad \text{and when } t = 0:$$

$$\vec{T} = \frac{a\vec{j} + b\vec{k}}{\sqrt{a^2+b^2}}$$

$$N = -i$$

$$B = \frac{-b\vec{j} + a\vec{k}}{\sqrt{a^2+b^2}}.$$

So, putting things in terms of the space frame, $\vec{T}$, $\vec{N}$, $\vec{B}$, we have:

$$\vec{i} = -\vec{N}.$$

$\vec{j}$ is a unit vector $\vec{v}$ satisfying:

$$\vec{v}\cdot\vec{T} = a/\sqrt{a^2+b^2}; \qquad \vec{v}\cdot\vec{N} = 0; \qquad \vec{v}\cdot\vec{B} = -b/\sqrt{a^2+b^2}$$

and

$$\vec{k} = \vec{i}\times\vec{j} = -\vec{N}\times\vec{v}.$$

Therefore, the spiral can be described by:

$$\vec{R} = a\cos(t)(-\vec{N}) + a\sin(t)(\vec{v}) + bt(-\vec{N}\times\vec{v}).$$

Finally, given a space curve and a point P on the curve and given $\vec{T}$, $\vec{N}$, κ and τ at P, and letting $\vec{P}$ be the vector joining the origin to point P, then the osculating spiral is given by:

$$\vec{R} = \vec{P} + (a\cos(t) - a)(-\vec{N}) + a\sin(t)(\vec{v}) + bt(-\vec{N}\times\vec{v})$$

where:

$$a = \frac{\kappa}{\kappa^2+\tau^2},\ b = \frac{\tau}{\kappa^2+\tau^2},\ \vec{v}\cdot\vec{T} = a/\sqrt{a^2+b^2} = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \qquad \vec{v}\cdot\vec{N} = 0$$

and

$$\vec{v}\cdot\vec{B} = -b/\sqrt{a^2+b^2} = \frac{-\tau}{\sqrt{\kappa^2+\tau^2}}$$

By construction this spiral has the same $\vec{T}$, $\vec{N}$, κ and τ as does the given curve at P and, in the sense noted before, no other approximation to the given curve near P is better, except, of course, the curve itself.

Notes

[1] The first half of this article is a distillation of the kind of development of $\vec{T}$, $\vec{N}$, and the osculating circle which you will find in any good calculus text such as *Calculus and Analytic Geometry* (6th ed.) by Thomas and Finney, Addison Wesley, 1984, Chap. 14, Sections 4, 5.

[2] I refer here to the Frenet formulas which show how the space frame, $\vec{T}$, $\vec{N}$, $\vec{B}$, varies with arclength. You can find the Frenet formulas derived in any good advanced calculus text such as *Advanced Calculus* by John M. H. Olmsted, Prentice-Hall, 1961, Chapter 19, Section 4. In the notation of this article we have:

$$d\vec{T}/ds = \kappa\vec{N}. \tag{1}$$

But $\vec{N}\cdot\vec{N}=1$; therefore, as in the argument given in the article, $d\vec{N}/ds$ must be perpendicular to $\vec{N}$ just as $d\vec{T}/ds$ is perpendicular to $\vec{T}$. This means that $d\vec{N}/ds$ must lie in the plane of $\vec{T}$ and $\vec{B}$. We have defined $\tau = d\vec{N}/ds\cdot\vec{B}$, so all we need now is the value of $d\vec{N}/ds\cdot\vec{T}$. Since $\vec{N}\cdot\vec{T}=0$, $d\vec{N}/ds\cdot\vec{T}+\vec{N}\cdot d\vec{T}/ds=0$. This says that $d\vec{N}/ds\cdot\vec{T}+\vec{N}\cdot\kappa\vec{N}=0$, i.e., $d\vec{N}/ds\cdot\vec{T}+\kappa=0$. Therefore:

$$d\vec{N}/ds = -\kappa\vec{T}+\tau\vec{B}. \tag{2}$$

Finally, $\vec{B}=\vec{T}\times\vec{N}$; therefore,

$$\begin{aligned} d\vec{B}/ds &= d\vec{T}/ds\times\vec{N}+\vec{T}\times d\vec{N}/ds = \kappa\vec{N}\times\vec{N}+\vec{T}\times(-\kappa\vec{T})+\vec{T}\times(\tau\vec{B}) \\ &= \vec{0}+\vec{0}+\tau(-\vec{N}), \end{aligned}$$

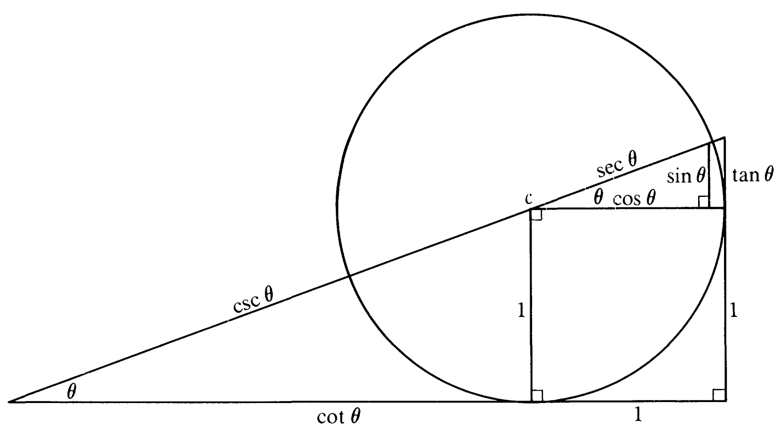
i.e.,

$$d\vec{B}/ds = -\tau\vec{N}. \tag{3}$$

Thus, curvature and torsion completely determine how the space frame, $\vec{T}$, $\vec{N}$, $\vec{B}$, varies with arclength and, hence, they completely determine the shape of the curve.

Proof without Words:

$$(\tan\theta+1)^2+(\cot\theta+1)^2=(\sec\theta+\csc\theta)^2$$



$$\tan^2\theta+1=\sec^2\theta$$
$$\cot^2\theta+1=\csc^2\theta$$
$$(\tan\theta+1)^2+(\cot\theta+1)^2=(\sec\theta+\csc\theta)^2$$
$$\left(\text{also}\quad \tan\theta=\frac{\tan\theta+1}{\cot\theta+1}\right)$$

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Irvington, NY

$$d\vec{T}/ds = \kappa\vec{N}. \tag{1}$$

But $\vec{N}\cdot\vec{N}=1$; therefore, as in the argument given in the article, $d\vec{N}/ds$ must be perpendicular to $\vec{N}$ just as $d\vec{T}/ds$ is perpendicular to $\vec{T}$. This means that $d\vec{N}/ds$ must lie in the plane of $\vec{T}$ and $\vec{B}$. We have defined $\tau = d\vec{N}/ds\cdot\vec{B}$, so all we need now is the value of $d\vec{N}/ds\cdot\vec{T}$. Since $\vec{N}\cdot\vec{T}=0$, $d\vec{N}/ds\cdot\vec{T}+\vec{N}\cdot d\vec{T}/ds=0$. This says that $d\vec{N}/ds\cdot\vec{T}+\vec{N}\cdot\kappa\vec{N}=0$, i.e., $d\vec{N}/ds\cdot\vec{T}+\kappa=0$. Therefore:

$$d\vec{N}/ds = -\kappa\vec{T}+\tau\vec{B}. \tag{2}$$

Finally, $\vec{B}=\vec{T}\times\vec{N}$; therefore,

$$\begin{aligned} d\vec{B}/ds &= d\vec{T}/ds\times\vec{N}+\vec{T}\times d\vec{N}/ds = \kappa\vec{N}\times\vec{N}+\vec{T}\times(-\kappa\vec{T})+\vec{T}\times(\tau\vec{B})\\ &= \vec{0}+\vec{0}+\tau(-\vec{N}), \end{aligned}$$

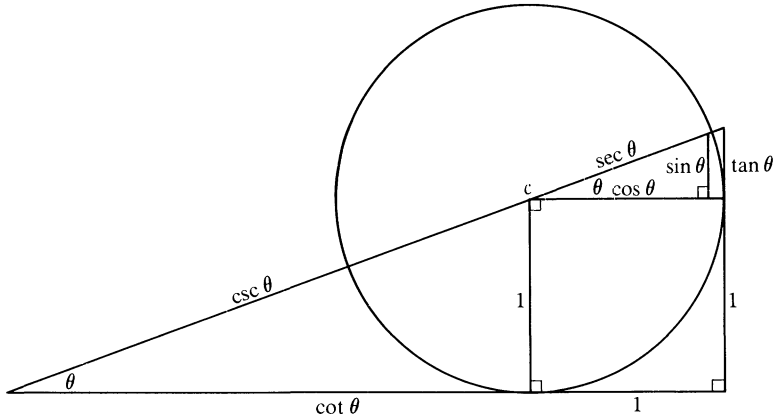
i.e.,

$$d\vec{B}/ds = -\tau\vec{N}. \tag{3}$$

Thus, curvature and torsion completely determine how the space frame, $\vec{T}$, $\vec{N}$, $\vec{B}$, varies with arclength and, hence, they completely determine the shape of the curve.

Proof without Words:

$$(\tan\theta+1)^2+(\cot\theta+1)^2=(\sec\theta+\csc\theta)^2$$



$$\tan^2\theta+1=\sec^2\theta$$
$$\cot^2\theta+1=\csc^2\theta$$
$$(\tan\theta+1)^2+(\cot\theta+1)^2=(\sec\theta+\csc\theta)^2$$
$$\left(\text{also}\quad \tan\theta=\frac{\tan\theta+1}{\cot\theta+1}\right)$$

—WILLIAM ROMAINE
Irvington, NY

PROBLEMS

LOREN C. LARSON, *editor*
St. Olaf College

BRUCE HANSON, *associate editor*
St. Olaf College

Proposals

To be considered for publication, solutions should be received by September 1, 1988.

1292. *Proposed by Murray S. Klamkin, University of Alberta, Canada.*

Determine the maximum value of

$$x_1^2x_2 + x_2^2x_3 + \cdots + x_n^2x_1,$$

given that $x_1 + x_2 + \cdots + x_n = 1$, $x_1, x_2, \ldots, x_n \geqslant 0$ and $n \geqslant 3$.

1293. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis.*

Let $\mathbf{A}$ be an $n \times n$ matrix with integer entries. Prove that $\gcd(\mathbf{AX}) = \gcd(\mathbf{X})$ for every integer column n-tuple $\mathbf{X}$ if and only if $\det(\mathbf{A}) = \pm 1$. ($\gcd(\mathbf{X})$ denotes the greatest common divisor of the entries of $\mathbf{X}$.)

1294. *Proposed by William Dunham, Hanover College, Indiana.*

Express $\int \sqrt{\tan x}\, dx$ in closed form.

1295. *Proposed by Edward Kitchen, Santa Monica, California.*

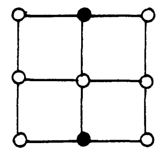
Let R be a given rectangle. Construct a square outwards on the length of R; construct another square outwards on the length of the resulting rectangle. Continue this process anticlockwise indefinitely.

a. Prove that the centers of the spiraling squares lie on two perpendicular lines.

b. As the process continues, show that the ratio of the sides of the rectangles approaches the golden mean.

1296. *Proposed by Howard Cary Morris, Xerox Computer Services, Los Angeles, California.*

Consider an $n \times n$ square lattice with points colored either black or white. A *square path* is a closed path in the shape of a square with edges parallel to the edges of the lattice. Let $M(n)$ be the minimum number of black points needed for an $n \times n$ square lattice so that every square path has at least one black point on it. (For example, $M(3) = 2$; see figure.)



a. Show that $\lim_{n \to \infty} M(n)/n^2$ exists.
b. Evaluate $\lim_{n \to \infty} M(n)/n^2$.
*c. Find a formula for $M(n)$.

Quickies

Answers to the Quickies are on pages 126–7.

Q731. *Submitted by Daniel Goffinet, Saint Étienne, France.*

Start with a finite sequence S_0. Generate a new sequence S_1 by replacing each term of S_0 by the number of times it appears in S_0. Continue in this way: S_{n+1} is formed by replacing each element of S_n by the number of times it appears in S_n. Will it always happen that for some integer n, $S_{n+1} = S_{n+2} = \cdots$?

Q732. *Proposed by Norman Schaumberger, Bronx Community College, New York.*

If p, q, and r are positive integers, show that

$$3^{p+q+r} \geqslant \left(1 + \frac{q+r}{p}\right)^p \left(1 + \frac{r+p}{q}\right)^q \left(1 + \frac{p+q}{r}\right)^r.$$

Q733. *Proposed by Murray S. Klamkin, University of Alberta.*

If A, B, C are angles of a triangle, determine the maximum area of a triangle whose sides are $\cos(A/2), \cos(B/2), \cos(C/2)$.

Solutions

Integer recurrence **February 1987**

1259. *Proposed by John P. Hoyt, Lancaster, Pennsylvania (as corrected).*

Given two nonzero integers a and b with $|a| \neq |b|$, define a sequence $(b_n)_{n=1}^{\infty}$ by the recurrence

$$b_1 = b, \qquad b_2 = b^2 - a^2, \qquad \text{and} \quad b_{k+2} = \frac{b_{k+1}^2 - a^{2(k+1)}}{b_k} \quad \text{for} \quad k \geq 1.$$

Show that b_n is an integer for all $n \geq 1$.

I. *Solution by John Oman, University of Wisconsin, Oshkosh.*

We begin by showing that b_k satisfies the recurrence relation

$$b_{k+2} = bb_{k+1} - a^2 b_k \quad \text{for} \quad k = 1,2,3,\ldots. \tag{1}$$

The proof is by induction. It is easily checked for $k = 1$. Assume the relation is valid for $k = 1,2,\ldots, n-1$. Then

$$\begin{aligned} b_{n+2} &= \frac{b_{n+1}^2 - a^{2(n+1)}}{b_n} = \frac{\left(bb_n - a^2 b_{n-1}\right)^2 - a^{2(n+1)}}{b_n} \\ &= b^2 b_n - 2a^2 bb_{n-1} + \frac{a^4 b_{n-1}^2 - a^{2(n+1)}}{b_n} \\ &= b^2 b_n - 2a^2 bb_{n-1} + \frac{a^4 b_{n-2}}{b_n} \frac{b_{n-1}^2 - a^{2(n-1)}}{b_{n-2}} \\ &= b^2 b_n - 2a^2 bb_{n-1} + a^4 b_{n-2} \\ &= b\left(bb_n - a^2 b_{n-1}\right) - a^2\left(bb_{n-1} - a^2 b_{n-2}\right) \\ &= bb_{n+1} - a^2 b_n. \end{aligned}$$

From (1) it is obvious that b_k is an integer for each k. For the above to be valid it remains to show that b_k is never 0. Temporarily denote the sequence by $b_k(a, b)$ to emphasize the dependence on a and b. It is easily verified that

(i) $b_k(-a, b) = b_k(a, b)$;

(ii) $b_k(ra, rb) = r^k b_k(a, b)$ for any nonzero real number r;

(iii) $b_k(a, -b) = (-1)^k b_k(a, b)$.

Thus we need only prove that b_k is nonzero if a and b are positive and relatively prime.

LEMMA 1. *If $b \geq a^2 + 1$, then b_k is a strictly increasing sequence of positive integers.*

$b_2 = b^2 - a^2 \geq b(a^2 + 1) - a^2 = b + a^2(b - 1) > b = b_1 > 0$. Assume that $b_k > b_{k-1}$ for $k = 2,3,\ldots, n$. Then $b_{n+1} = bb_n - a^2 b_{n-1} > bb_n - a^2 b_n > b_n$, and the proof is complete by induction.

LEMMA 2. *If a and b are relatively prime and a is greater than 1, then a does not divide b_k for any k.*

Again the proof is by induction. a does not divide b_1 since it is relatively prime to b $(= b_1)$.

Assume that a divides b_2. Then a divides $b^2 = b_2 + a^2$. But this is impossible since a and b are relatively prime.

Assume that a does not divide b_k for $k = 1,2,\ldots, n-1$. If a were to divide b_n then a would divide $bb_{n-1} = b_n + a^2 b_{n-2}$. But a and b are relatively prime and a does not divide b_{n-1} by the induction hypothesis. Thus a does not divide b_k for any k.

If $a = 1$ then $b \geq a^2 + 1$ since $|a| \neq |b|$, and Lemma 1 implies that $b_k \neq 0$. If $a > 1$ then Lemma 2 implies that a does not divide b_k so in particular, $b_k \neq 0$.

II. *Solution by Lorraine L. Foster, California State University at Northridge.*
Let Q and Z denote the sets of rational numbers and integers respectively.

THEOREM. *Let $\{f_n\} \subset Q(x)$ be defined recursively by*

$$(1) \quad f_0(x)=1,\ f_1(x)=x,\ \text{and}\ \ f_{n+2}(x)=\frac{f_{n+1}^2(x)-1}{f_n(x)}, \quad \text{for} \quad n \geq 0.$$

Then, for $m \geq 1$ and $n \geq 0$ we have:

(2) $f_{2m} = f_m^2 - f_{m-1}^2$,

(3) $f_{2m+1} = f_m(f_{m+1} - f_{m-1})$,

(4) $f_n \in Z[x]$, *f_n is monic and* $\deg f_n = n$. *Furthermore* $|f_n(0)| = 1$ *or* 0 *according as n is even or odd, and, if $f_n(r) = 0$ for $r \in Q$, then $|r| = 0$ or 1.*

Proof. It is straightforward to check that (2) and (3) hold for $m = 1$, and (4) holds for $0 \leq n \leq 3$. Assume the truth of (2), (3), and (4) for all m such that $1 \leq m \leq k$ and all n such that $0 \leq n \leq 2k+1$. From (1) and (4), $(f_k^2 - f_{k+1}f_{k-1})^2 = 1$, so that $f_k^4 + f_{k+1}^2 f_{k-1}^2 = 2f_k^2 f_{k+1} f_{k-1} + 1$. Hence, by (2), (3), and (1), one shows that $(f_{k+1}^2 - f_k^2)f_{2k} = f_{2k+2}f_{2k}$ so that by (4), $f_{k+1}^2 - f_k^2 = f_{2k+2}$. Hence (2) is true for $m = k+1$. With a little more manipulation, one shows that $f_{2k+2}^2 - 1 = f_{2k+1}(f_{k+1}(f_{k+2} - f_k))$, so that $f_{2k+3} = (f_{2k+2}^2 - 1)/f_{2k+1} = f_{k+1}(f_{k+2} - f_k)$, and therefore (3) also is true for $m = k+1$. It follows easily from (2) and (3) that f_n is a monic integral polynomial of degree n for $n = 2k+2$ and $2k+3$. Also, $f_{2k+2}(0) = \pm 1$ and $f_{2k+3}(0) = 0$. Since f_n is monic and integral for $n \leq 2k+3$, we know that if $f_n(r) = 0$ and $r \in Q$, then in fact $r \in Z$. Further, if $f_{2k+2}(r) = 0$ for $r \in Z$, then $|r| = 1$. If $f_{2k+3} = 0$ for some $r \in Z$ such that $|r| \neq 0, 1$, then $f_{k+1}(r)(f_{k+2}(r) - f_k(r)) = 0$, and therefore $f_{k+2}(r) = f_k(r)$. It follows that $f_{k+1}^2(r) - 1 = f_{k+2}(r)f_k(r) = f_k^2(r)$, so that (since $f_k(r) \in Z$), $f_k(r) = 0$, a contradiction. Hence (4) holds for $n = 2k+2$ and $2k+3$ and the induction is complete.

The proposed result is obtained by defining $b_k = a^k f_k(b/a)$ for all $k \geq 0$. (From (4), $a^k f_k(b/a)$ is a nonzero integer for all pairs of integers a and b such that $ab \neq 0$ and $|a/b| \neq 1$).

Also solved by Robert E. Bernstein, David Callan, Farhood Pouryoussefi (student, Iran), J. M. Stark, and the proposer.

The solution by *Callan* makes use of the fact that the polynomials defined in solution II by Foster are closely related to the Tchebychev polynomials of the second kind.

Minimal cycle representation **February 1987**

1260. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, Maryland.*

Among all ways of writing a given permutation in S_n as a product of cycles, does the disjoint cycle representation use the fewest number of nontrivial cycles? If so, give a proof; if not, describe a correct minimal representation.

Solution by the Chico Problem Group, California State University, Chico.

No. Every permutation is either a cycle or a product of two cycles. To see this, observe that if

$$\alpha = (a_{11}, a_{12}, \ldots, a_{1k_1})(a_{21}, a_{22}, \ldots, a_{2k_2}) \cdots (a_{r1}a_{r2}, \ldots, a_{rk_r})$$

is a representation of α as a product of disjoint cycles, $r > 1$, then

$$\alpha=(a_{11},\ldots,a_{1k_1},a_{21},\ldots,a_{2k_2},\ldots,a_{r1},\ldots,a_{rk_r})(a_{r1},\ldots,a_{21},a_{11}).$$

Also solved by Hamza Yousef Ahmad (student, Kuwait), S. F. Barger, David Callan, M. G. Deshpande, Emil F. Knapp, Gary L. Walls, John T. Ward, and the proposer.

Integral triangles and tetrahedrons **February 1987**

1261. *Proposed by Stanley Rabinowitz, Digital Equipment Corporation, Nashua, New Hampshire.*

a. What is the area of the smallest triangle with integral sides and integral area?
*b. What is the volume of the smallest tetrahedron with integral sides and integral volume?

(a) *Solution by Murray S. Klamkin, University of Alberta.*

We will show that the area of a triangle with integral sides and integral area is divisible by 6. Since a 3-4-5 right triangle has area 6, the smallest area is necessarily 6.

Let T be a triangle with integral sides, a, b, c, and integral area. We may assume that $\gcd(a,b,c)=1$. Let s denote the semiperimeter.

The inradius $r=\text{Area}/s$ and thus must be rational. Since $\tan(A/2)=r/(s-a)$, $\tan(A/2)$ is rational and let it equal n/m where $(m,n)=1$. Similarly, $\tan(B/2)=q/p$, with $(p,q)=1$. Then

$$\sin A=\frac{2mn}{(m^2+n^2)},\qquad \cos A=\frac{(m^2-n^2)}{(m^2+n^2)},\qquad \sin B=\frac{2pq}{(p^2+q^2)},\quad \text{etc.,}$$

and

$$\sin C=\sin(A+B)=\frac{2(mq+np)(mp-nq)}{(m^2+n^2)(p^2+q^2)}.$$

The sides of any triangle are proportional to the sines of the opposite angles; specifically, $a=2R\sin A$, etc., where R is the circumradius. Let us take $4R=(m^2+n^2)(p^2+q^2)$, and let $\bar{T}$ be the triangle with sides

$$\begin{aligned}\bar{a}&=mn(p^2+q^2),\qquad \bar{b}=pq(m^2+n^2),\\ \bar{c}&=(mq+np)(mp-nq)=mn(p^2-q^2)+pq(m^2-n^2).\end{aligned}\tag{1}$$

We may assume that mn is relatively prime to pq, otherwise we can divide out the common factor. $\bar{T}$ has integral area (Area $\bar{T}=\frac{1}{2}\bar{b}\bar{c}\sin A$), and $\bar{T}$ is similar to T. Also,

$$\text{Area }\bar{T}=\frac{\bar{a}\bar{b}\bar{c}}{4R}=mnpq\big(mn(p^2-q^2)+pq(m^2-n^2)\big).\tag{2}$$

Suppose that $\bar{a}=da$, $\bar{b}=db$, $\bar{c}=dc$, for integer $d\geq 1$. Then, Area $\bar{T}=d^2$Area T.

Suppose d is even. Then, from (1) and our supposition that $\gcd(m,n,p,q)=1$, it must be the case that m,n,p,q are odd. In this case $\bar{a},\bar{b},\bar{c}$ are divisible by 2 but not by 4; i.e., d is not divisible by 4. Also, Area $\bar{T}$ is divisible by 8 (in (2), $p^2-q^2\equiv m^2-n^2\equiv 0 \pmod 8$), and therefore Area T is even.

If one of m,n,p,q is even the sides do not have 2 as a common factor, but Area $\bar{T}$, and hence Area T, is still even.

Equation (1), together with $\gcd(m,n,p,q)=1$, shows that 3 is not a common factor of $\bar{a},\bar{b},\bar{c}$; that is, d is not a multiple of 3.

If one of m, n, p, q is divisible by 3, then the sides are not divisible by 3, but Area $\bar{T}$ (and Area T) is. If none of m, n, p, q is divisible by 3, Area $\bar{T}$ (and Area T) is because $p^2 - q^2 \equiv m^2 - n^2 \equiv 0 \pmod 3$.

Thus, Area T is divisible by 6, and the proof is complete.

Part (a) also solved by Nicolas K. Artemiadis (Greece), Seung Jin Bang (Korea), Milton P. Eisner, Francis M. Henderson, David W. Koster, L. Kuipers (Switzerland), Mike Pinter, William P. Wardlaw, and the proposer. There was one incorrect solution and one incomplete solution to Part (a).

No solutions were received for Part (b). The proposer supplied computer generated evidence (a list of tetrahedra with small volumes) that suggests that the smallest volume is 6. Also, see *Crux Mathematicorum* (May 1985) 162–166, for a consideration of tetrahedra having integer-valued edge lengths, face areas, and volume.

Koster showed that the problem is equivalent to finding the smallest triangle with integer sides and *rational* area.

Enumeration of rationals **April 1987**

1262. *Proposed by Erwin Just, Bronx Community College, Bronx, New York.*

Show that it is possible to enumerate the rational numbers in the open interval (0, 1) so that in their decimal expansions

$$r_1 = .a_{11}a_{12}a_{13}a_{14}\cdots$$
$$r_2 = .a_{21}a_{22}a_{23}a_{24}\cdots$$
$$r_3 = .a_{31}a_{32}a_{33}a_{34}\cdots$$
$$\cdots$$

the "columns" $c_k = .a_{1k}a_{2k}a_{3k}a_{4k}\cdots$ are rational for $k = 1, 2, 3\dots$.

Solution by G. A. Heuer, Concordia College, Minnesota.

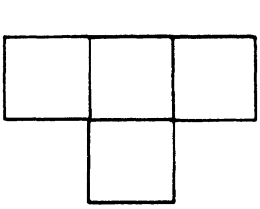
Let $s_1, s_2, s_3, \dots$ be an enumeration of the rational numbers in (0, 1). For $1 \le i \le 10$ let r_i be the first s_k with first digit $i - 1$. For $i = 10 + j$, $1 \le j \le 10^2$, let r_i be the first previously unselected s_k with first two digits ba, where $j - 1 = 10a + b$. For $i = 10 + 10^2 + j$, $1 \le j \le 10^3$, let r_i be the first previously unselected s_k with first three digits cba, where $j - 1 = 10^2a + 10b + c$, and so forth. Clearly, every s_k is eventually selected. In the first column we have the repeating cycle $012\cdots9$; in the second column after the first ten digits the repeating cycle of length 100 consisting of ten 0's, ten 1's, ..., ten 9's, and so forth.

Also solved by Kenneth L. Bernstein, Irl C. Bivens, Jerrold W. Grossman, Ricardo Perez Marco (student, Spain), Elliott A. Weinstein, Western Maryland College Problems Group, and the proposer.

Tiling with T-tetrominoes **April 1987**

1263. *Proposed by Miklós Laczkovich, Eötvös Lorand University, Budapest, Hungary.*

A T-tetromino is a configuration of four unit squares arranged in the following shape:



a. Prove that if an $m \times n$ rectangular board can be tiled with T-tetrominoes, then the product mn is a multiple of eight.

*b. Prove or disprove: An $m \times n$ rectangular board can be tiled with T-tetrominoes if and only if m and n are multiples of four.

(a) *Solution by Jerrold W. Grossman, Oakland University, Michigan.*

Let a tiling be given. Since the T-tetromino has 4 squares, mn must be a multiple of 4. Without loss of generality assume that each row has even length. Imagine the board as a red and white checkerboard. Since half the squares in each row are red, the number of red squares in the entire board equals the number of white squares.

Now each T-tetromino covers either 3 red squares and 1 white square, or 1 red square and 3 white squares. If there were an unequal number of T-tetrominoes of these two types, then the number of red squares covered would not be equal to the number of white squares covered, a contradiction. It follows that there are an even number of T-tetrominoes altogether (each covering 4 squares), and hence mn is a multiple of 8.

Part (a) also solved by Robert E. Bernstein, Richard A. Gibbs, J. Heuver (Canada), Richard Johnsonbaugh, J. C. Linders (The Netherlands), Ricardo Perez Marco (student, Spain), Nick Martin (student), William P. Wardlaw, and the proposer. There was one incorrect solution.

J. L. Selfridge and Edward T. H. Wang note that a proof of Part (b) is given by D. W. Walkup in Covering a rectangle with T-tetrominoes, *Amer. Math. Monthly* (1965), 986–988.

Convergence of averages **April 1987**

1264. *Proposed by Daniel B. Shapiro and Bostwick Wyman, The Ohio State University, Columbus.*

a. Given $a_1, a_2 \in \mathbf{R}$ define the sequence $(a_k)_{k=1}^{\infty}$ by setting

$$a_k = \frac{a_{k-1} + a_{k-2}}{2}$$

for all $k \geq 3$. Prove that $L(a_1, a_2) \equiv \lim_{k \to \infty} a_k$ exists and equals

$$\tfrac{1}{3}a_1 + \tfrac{2}{3}a_2.$$

b. More generally, for fixed n, let $a_1, a_2, \ldots, a_n \in \mathbf{R}$ be given and define the sequence $(a_k)_{k=1}^{\infty}$ by setting

$$a_k = \frac{a_{k-1} + a_{k-2} + \cdots + a_{k-n}}{n}$$

for $k \geq n+1$. Prove that $L(a_1, a_2, \ldots, a_n) \equiv \lim_{k \to \infty} a_k$ exists and that L is a linear function on R^n. That is, show that there exist constants $c_j \in R$ such that $L(a_1, a_2, \ldots, a_n) = c_1 a_1 + c_2 a_2 + \cdots + c_n a_n$. Evaluate these constants c_j.

I. *Solution by David Siegel, University of Waterloo, Canada.*

Consider the general case, part (b). We first show that the limit exists. To this end, let

$$U_k = \max\{a_{k-1}, a_{k-2}, \ldots, a_{k-n}\} \quad \text{and} \quad L_k = \min\{a_{k-1}, a_{k-2}, \ldots, a_{k-n}\}.$$

Since

$$a_k = \frac{a_{k-1} + a_{k-2} + \cdots + a_{k-n}}{n} \le U_k,$$

we have

$$U_{k+1} = \max\{a_k, a_{k-1}, \dots, a_{k-n+1}\} \le U_k.$$

Similarly, $L_{k+1} \ge L_k$. Clearly, $U_k \ge L_k$, so that (U_k) and (L_k) are monotone bounded sequences which therefore have limits U and L, respectively, with $U \ge L$. We claim that $U = L$. By the definition of U, for $\varepsilon > 0$, there exists an integer N such that $U_k < U + \varepsilon$ for $k \ge N$. Note that since one of $a_{k-1}, \dots, a_{k-n}$ is equal to L_k,

$$a_k = \frac{a_{k-1} + \cdots + a_{k-n}}{n} \le \frac{L_k + (n-1)U_k}{n},$$

and therefore,

$$a_k < \frac{L + (n-1)(U+\varepsilon)}{n}$$

for $k \geqslant N$. This implies that

$$U_{N+n} = \max\{a_{N+n-1}, \dots, a_N\} < \frac{L + (n-1)(U+\varepsilon)}{n}.$$

Hence

$$U < \frac{L + (n-1)(U+\varepsilon)}{n},$$

or equivalently, $U - L < (n-1)\varepsilon$. Since ε is arbitrary, $U - L \le 0$. It follows that $U = L$. Clearly $\lim_{n \to \infty} a_n$ exists and is equal to this common limit.

Next we evaluate the limit. By adding the equations

$$\begin{aligned}
a_1 + a_2 + a_3 + \cdots + a_n &= na_{n+1} \\
a_2 + a_3 + \cdots + a_n + a_{n+1} &= na_{n+2} \\
a_3 + \cdots + a_n + a_{n+1} + a_{n+2} &= na_{n+3} \\
\cdots & \\
a_n + a_{n+1} + \cdots + a_{n+(n-1)} &= na_{n+n}
\end{aligned}$$

and "telescoping" we have

$$a_1 + 2a_2 + 3a_3 + \cdots + na_n = a_{n+1} + 2a_{n+2} + \cdots + na_{n+n}.$$

Repeating the argument gives

$$a_1 + 2a_2 + \cdots + na_n = a_{pn+1} + 2a_{pn+2} + \cdots + na_{pn+n}$$

for any positive integer p. Now let p tend to infinity and we get

$$a_1 + 2a_2 + \cdots + na_n = (1 + 2 + \cdots + n)L = \frac{n(n+1)}{2}L.$$

Thus,

$$L = \frac{2}{n(n+1)}(a_1 + 2a_2 + \cdots + na_n).$$

II. *Solution by E. Lee, Boeing Commercial Airplane Company, Seattle, Washington.*

It suffices to consider part (b). We will prove a more geometrical reformulation: Let $A_1, \dots, A_n$ form an $(n-1)$-simplex in R^{n-1}. Let A_k be the centroid of $A_{k-1}, \dots, A_{k-n}$, $k > n$. Then

$$\lim_{k\to\infty} A_k = A = \frac{A_1 + 2A_2 + \cdots + nA_n}{1+2+\cdots+n}. \tag{1}$$

Proof. In this formulation, the convergence is obvious, because the successive simplices $\sigma_j = \sigma(A_{j+1}, \dots, A_{j+n})$, $j \geq 0$, are nested and $\mathrm{Vol}(\sigma_j) \to 0$ since $\mathrm{Vol}(\sigma_{j+1}) = \mathrm{Vol}(\sigma_j)/n$.

The limit A, being in σ_0, can be expressed as

$$A = c_1A_1 + \cdots + c_nA_n, \qquad c_1 + \cdots + c_n = 1.$$

Note that $c_1, \dots, c_n$ are independent of $A_1, \dots, A_n$. (The barycentric coordinates of the centroid are the same for any simplex, and thus the $c_{i,k}$'s in $A_k = c_{1,k}A_1 + \cdots + c_{n,k}A_n$ are independent of $A_1, \dots, A_n$. So must also be the c_i's, which are the limits of the $c_{i,k}$'s.) In particular, if we start out with $A_2, \dots, A_{n+1}$, the successive centroids converge to

$$c_1A_2 + \cdots + c_{n-1}A_n + c_n\left(\frac{A_1 + \cdots + A_n}{n}\right).$$

But the limit is the same no matter which stage we start from, so

$$c_1A_1 + \cdots + c_nA_n = \frac{c_n}{n}A_1 + \left(c_1 + \frac{c_n}{n}\right)A_2 + \cdots + \left(c_{n-1} + \frac{c_n}{n}\right)A_n.$$

Since $A_1, \dots, A_n$ are affinely independent and $c_1 + \cdots + c_n = 1$, we can equate coefficients. Thus

$$c_1 = c_n/n,$$
$$c_2 = c_1 + c_n/n = 2c_1,$$
$$c_3 = c_2 + c_n/n = 3c_1,$$

and so forth, proving (1).

The connection with the original problem is clear. If $a_1, \dots, a_n$ are not all equal, one can "lift" them to an affinely independent set $A_1, \dots, A_n$ with first coordinate in A_i equal to a_i. (For instance if $a_1 \neq a_2$, one can take $A_1 = (a_1, 0, \dots, 0)$, $A_2 = (a_2, 0, \dots, 0)$, $A_3 = (a_3, 1, 0, \dots, 0), \dots, A_n = (a_n, 0, \dots, 0, 1)$.) The first coordinates of (1) then yield the desired solution, which also holds trivially in case $a_1 = \cdots = a_n$.

III. *Solution by Y. H. Harris Kwong, SUNY, College at Fredonia, New York.*

Define the generating function

$$A(x) = \sum_{k=1}^{\infty} a_k x^k.$$

Multiply each side by $\dfrac{x + x^2 + \cdots + x^n}{n}$, simplify using the recurrence relation, and solve for $A(x)$. This yields

$$A(x) = \frac{\sum_{j=1}^{n}\left(a_j - \dfrac{a_1 + \cdots + a_{j-1}}{n}\right)x^j}{F(x)},$$

where

$$F(x)=1-\frac{1}{n}\sum_{j=1}^{n}x^{j}.$$

Let $1/F(x)=\sum_{k=0}^{\infty}b_kx^k$. Then $L(0,\dots,0,1)=\lim_{k\to\infty}b_k$. We shall show that $\lim_{k\to\infty}b_k=2/(n+1)$. It then follows that

$$\lim_{k\to\infty}a_k=\frac{2}{n+1}\sum_{j=1}^{n}\left(a_j-\frac{a_1+\cdots+a_{j-1}}{n}\right)=\sum_{j=1}^{n}\frac{2j}{n(n+1)}a_j.$$

Let

$$f(z)=z^nF(1/z)=z^n-\frac{z^{n-1}+\cdots+z+1}{n}$$

and

$$g(z)=\frac{z^n+\cdots+z+1}{n}.$$

Then $f(z)=(z-1)g'(z)$ since

$$g'(z)=z^{n-1}+\frac{(n-1)z^{n-2}+\cdots+2z+1}{n}.$$

From the Gauss-Lucas theorem (see, for example, Herbert S. Wilf, *Mathematics for the Physical Sciences*, John Wiley & Sons (1978), pp. 84–85), the zeros (over C) of $g'(z)$ lie in the convex hull, K, of the zeros (over C) of $g(z)$. Clearly, K is an n-gon whose vertices are the n distinct $(n+1)$th roots of unity, ω_i, where $\omega_i\neq 1, 1\le i\le n$. Since $g(z)$ and $g'(z)$ have no common zeros, none of the vertices of K can be a zero of $g'(z)$. Thus, the zeros of $g'(z)$ all lie in $|z|<1$. Furthermore, if we write $g(z)$ as $(z^{n+1}-1)/n(z-1)$, we have

$$\begin{aligned}g'(z)&=\frac{1}{n}\frac{(z-1)(n+1)z^n-(z^{n+1}-1)}{(z-1)^2}\\&=\frac{1}{n}\frac{(n+1)z^n-(z^n+z^{n-1}+\cdots+1)}{z-1}\\&=\frac{1}{n}\frac{(n+1)z^n-ng(z)}{z-1}.\end{aligned}$$

Then,

$$g''(z)=\frac{(z-1)\left[(n+1)nz^{n-1}-ng'(z)\right]+(n+1)z^n-ng(z)}{n(z-1)^2}.$$

If $\theta\neq 1$ and $g'(\theta)=g''(\theta)=0$, we have $(n+1)\theta^n-ng(\theta)=0$ and $(n+1)n\theta^{n-1}=0$, which is impossible. We conclude that $g'(z)$, and hence $f(z)$, have no repeated zeros.

Therefore, $F(z)=z^nf(1/z)=(1-z)\prod_{i=1}^{n-1}(1-\beta_iz)$, where β_i are distinct, $|\beta_i|<1, 1\le i\le n-1$, and

$$\frac{1}{F(z)}=\frac{\alpha}{1-z}+\sum_{i=1}^{n-1}\frac{\alpha_i}{1-\beta_iz}$$

for some $\alpha, \alpha_i \in C, 1 \le i \le n-1$. Then $b_k = \alpha + \sum_{i=1}^{n-1} \alpha_i \beta_i^k$. Since $|\beta_i| < 1$ for $1 \le i \le n-1$, $\lim_{k \to \infty} b_k = \alpha = -1/F'(1) = 2/(n+1)$, which is exactly what we wanted to prove.

Also solved by Seung Jin Bang (Korea), Jeff Benedict, Daniel J. Bernstein, Ginger Bolton and David R. Stone, Paul Bracken (Canada), Duane Broline, Chico Problem Group, L. Matthew Christophe, Jr., David Doster, Robert Doucette, Mordechai Falkowitz, Jayanthi Ganapathy and V. Ganapathy, G. A. Heuer, J. Heuver (Canada), Thomas Jager, Hans Kappus (Switzerland), Murray S. Klamkin (Canada), Benjamin G. Klein (two solutions), Václav Konečný, Kee-wai Lau (Hong Kong), F. W. Lemire (Canada), Hsiuhsiang Li, Anne L. Ludington, Gian Carlo Mangano, Nick Martin (student; two solutions), David Morin (student), H. G. Mushenheim, William A. Newcomb, Gene M. Ortner, Harry D. Ruderman, Harvey Schmidt, Jr., Allen J. Schwenk, Harry Sedinger, Jan Söderkvist (Sweden), S. Thompson, William P. Wardlaw, Western Maryland College Problem Group (two solutions), Staffan Wrigge (Sweden), A. Zulauf (New Zealand; three solutions), and the proposers. There was one unsigned solution.

Part (a) only was solved by Farid G. Bassiri, Ragnar Dybvik (Norway), Enzo R. Gentile (Argentina), M. R. Gopal, María Luisa Oliver (Argentina), Volkhard Schindler (East Germany), and Edward T. H. Wang (Canada, two solutions).

Several people noted that part (a) is well known in the literature. For example, it appears as problem E1042 in *The American Mathematical Monthly* (1953), 420, and at the time, was ranked as the third most popular problem ever to appear in the *Monthly* (based on the number of solutions received). Jim Gowers, Graham Davis, and Elizabeth Miles, give two solutions to Part (b) (one using matrix algebra and one via difference equations) in Sequences of averages, *The Mathematical Gazette* (1986), 200-203.

Several people considered the more general case, $a_k = w_1 a_{k-1} + \cdots + w_n a_{k-n}$, where $w_1 + \cdots + w_n = 1$ and $w_1, \dots, w_n > 0$. The Chico Problem Group allowed the weighting factors to be non-negative, $w_1, \dots, w_n \ge 0$, $w_n > 0$, and found necessary and sufficient conditions for convergence.

L. M. Christophe, Jr., also studied the recursions

(i) $$a_{n+1} = \sqrt[k]{\prod_{j=0}^{k-1} a_{n-j}} \qquad (n \ge k, a_j > 0);$$

(ii) $$a_{n+1} = \frac{k \prod_{j=0}^{k-1} a_{n-j}}{\sum_{i=0}^{k-1} \left(\prod_{\substack{j=0 \\ j \ne i}}^{k-1} a_{n-j} \right)} \qquad (n \ge k);$$

(iii) $$a_{n+1} = \sqrt{\frac{1}{k} \sum_{i=0}^{k-1} a_{n-i}^2} \qquad (n \ge k),$$

and found the limits

(i) $$\sqrt[k(k+1)]{\prod_{j=1}^{k} a_j^{2j}};$$

(ii) $$\frac{k+1}{2} \left\{ \frac{k \prod_{j=1}^{k} a_j}{\sum_{i=1}^{k} \left(i \prod_{\substack{j=1 \\ j \ne i}}^{k} a_j \right)} \right\};$$

(iii) $$\sqrt{\frac{2}{k(k+1)} \sum_{i=1}^{k} i a_i^2},$$

respectively.

Orthomedian triangles **April 1987**

1265. *Proposed by M. S. Klamkin, University of Alberta, Canada.*

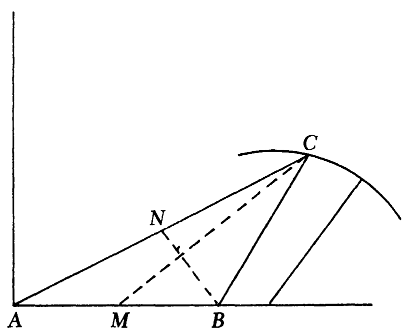
Determine the maximum area F of a triangle ABC if one side is of length λ and two of its medians intersect at right angles.

I. *Solution by Cornelius Groenewoud, Bartow, Florida.*

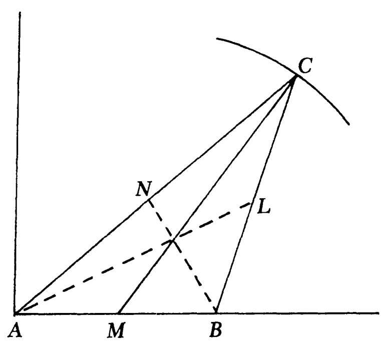
We show that $F = 3\lambda^2/8$ or $F = 3\lambda^2/4$ depending on whether one of the perpendicular medians is drawn to the side of length λ or not.

Two cases are considered: (i) One of the perpendicular medians is drawn to the side AB of length λ; (ii) Neither of the perpendicular medians is drawn to AB.

Place the side AB along the positive x-axis with A at the origin of a rectangular coordinate system. For the first case let M be the midpoint of AB and N the midpoint of AC.



Case i



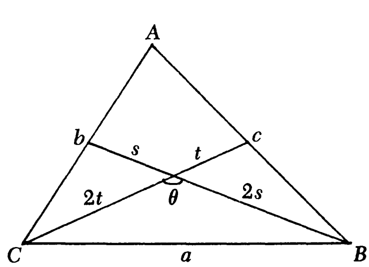
Case ii

Imposing the negative reciprocal condition between the slopes of medians BN and CM shows that the medians to AB and AC are perpendicular when C lies on a circle of radius $3\lambda/4$, centered at $P(5\lambda/4, 0)$. The area, $\lambda h/2$, will be maximum when h attains its maximum of $3\lambda/4$. For this case $F = 3\lambda^2/8$.

For the second case let L be the midpoint of BC. Imposing the negative reciprocal condition on the slopes of the medians AL and BN shows that the medians to AC and BC are perpendicular if C lies on a circle of radius $3\lambda/2$ centered at the midpoint of AB. The area is $\lambda h/2$, but in this case the maximum value of h is $3\lambda/2$. The corresponding area is $F = 3\lambda^2/4$.

II. *Solution by Thomas Jager, Calvin College, Michigan.*

The maximum is $3\lambda^2/4$. Label triangle ABC as shown, where the medians from B and C intersect in an angle θ.



By the law of cosines, $a^2 = 4t^2 + 4s^2 - 8st\cos\theta$, $(c/2)^2 = t^2 + 4s^2 - 4st\cos(\pi - \theta)$, and $(b/2)^2 = s^s + 4t^2 - 4st\cos(\pi - \theta)$. These equations imply that

$5a^2 - b^2 - c^2 = -8st\cos\theta$. It follows that the medians from B and C intersect in a right angle if and only if $5a^2 = b^2 + c^2$.

Suppose $\theta = \pi/2$. By the law of cosines, $a^2 = b^2 + c^2 - 2bc\cos A$, and thus,

$$F^2 = \left(\tfrac{1}{2}bc\sin A\right)^2 = \tfrac{1}{4}b^2c^2 - \tfrac{1}{4}b^2c^2\cos^2 A = \tfrac{1}{4}b^2c^2 - \tfrac{1}{4}\left(\frac{b^2+c^2-a^2}{2}\right)^2 = \tfrac{1}{4}b^2c^2 - a^4.$$

If $a = \lambda$, $F^2 = \frac{1}{4}b^2(5\lambda^2 - b^2) - \lambda^4$, and this is maximized when $b^2 = 5\lambda^2/2$, giving $F = 3\lambda^2/4$.

If $b = \lambda$, $F^2 = \frac{1}{4}\lambda^2(5a^2 - \lambda^2) - a^4 = \frac{9}{64}\lambda^4 - (a^2 - \frac{5}{8}\lambda^2)^2$, is maximized when $a = 5\lambda^2/8$, giving $F = 3\lambda^2/8$.

Also solved by Anders Bager (Denmark), Farid G. Bassiri (student), J. M. Bossert, Ragnar Dybvik (Norway), David Earnshaw (Canada), Peter Flanagan-Hyde, Chuck Hixon, Geoffrey A. Kendall, Hans Kappus (Switzerland), V. Konečný and R. Shepler, J. C. Linders (The Netherlands), Helen M. Marston, Ricardo Perez Marco (student, Spain), Edmond N. Mullins (student), William A. Newcomb, Al Nicholson (Canada), Farhood Pouryoussefi (student, Iran), M. M. Parmenter and B. L. R. Shawyer (Canada), Maroof A. Quidwai (Pakistan), Vidhyanath K. Rao, Francisco Bellot Rosado (Spain), Sahib Singh, J. M. Stark, László Szücs, Jan Söderkvist (Sweden), Garrett R. Vargas, Michael Vowe (Switzerland), Harry Weingarten, and the proposer. There were three incorrect solutions.

Groups as even permutations **April 1987**

1266. *Proposed by John W. Goppelt, Haverford, Pennsylvania.*

a. Prove that every finite group is isomorphic to a group of even permutations.

b. Let G be a finite group and H a subgroup of index two in G. Prove that there is a group of permutations isomorphic to G whose even permutations correspond to H.

Solution by Allen J. Schwenk, Western Michigan University.

(a) It is a standard exercise that every finite group can be realized as a group G of permutations. If G has both odd and even permutations, the even ones form a subgroup H of index two. Select two new symbols, say x and y, and add the transposition (x, y) to each odd permutation. We have not changed the multiplication table for G since any product that gave an odd permutation before will now have an extra transposition (x, y) whereas whenever the previous product was even the factors (x, y) will cancel.

(b) First realize G as in part (a) with only even permutations. Select two new symbols, say w and z, and insert a new transposition (w, z) on the elements *not* in H. As in part (a), this produces the desired parity without altering the multiplication table.

Also solved by Anders Bager (Denmark), J. M. Bossert, Duane Broline, Lorraine L. Foster, Enzo R. Gentile (Argentina), Albert F. Gilman III, Dennis Hamlin (student), Thomas Jager, David W. Koster, Francis C. Leary, Willard L. Maier, David E. Manes, Vidhyanath K. Rao, Harvey Schmidt, Jr., Gary L. Walls, William P. Wardlaw, and the proposer.

Answers

Solutions to the Quickies on p. 115.

A731. Yes, in every case, the sequence (S_n) will eventually stabilize. To see this, let t_n denote the number of distinct elements in S_n. Then $t_1 \geqslant t_2 \geqslant t_3 \geqslant \cdots$. Let n be the smallest integer such that $t_n = t_{n+1}$. Then no two distinct terms of S_n appear the same number of times. As a consequence, $S_{n+1} = S_{n+2} = \cdots$.

A732. The arithmetic mean-geometric mean inequality gives

$$\frac{3}{p+q+r} = \frac{\overbrace{\left(\frac{1}{p}+\cdots+\frac{1}{p}\right)}^{p\text{ times}} + \overbrace{\left(\frac{1}{q}+\cdots+\frac{1}{q}\right)}^{q\text{ times}} + \overbrace{\left(\frac{1}{r}+\cdots+\frac{1}{r}\right)}^{r\text{ times}}}{p+q+r}$$

$$\geqslant \left(\frac{1}{p^p q^q r^r}\right)^{\frac{1}{p+q+r}}.$$

Hence,

$$3^{p+q+r} \geqslant \frac{(p+q+r)^{p+q+r}}{p^p q^q r^r} = \left(1+\frac{q+r}{p}\right)^p \left(1+\frac{r+p}{q}\right)^q \left(1+\frac{p+q}{r}\right)^r.$$

A733. Comment: One might start from the expression for the square of the area of the triangle, i.e.,

$$F^2 = s\left(s-\cos\frac{A}{2}\right)\left(s-\cos\frac{B}{2}\right)\left(s-\cos\frac{C}{2}\right), \quad \text{where } 2s = \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2},$$

and then maximize subject to the constraints $A+B+C = 180°$ and $A, B, C \geqslant 0$.

Solution: First, one should verify that triangles with the given sides exist for all triangles ABC. If $A \geqslant B \geqslant C$, it suffices to show that

$$\cos\frac{A}{2}+\cos\frac{B}{2} > \cos\frac{C}{2},$$

or equivalently,

$$\cos\frac{A-B}{4} > \sin\frac{A+B}{4},$$

which follows.

An alternate proof of the existence of a triangle follows from the fact that the three sides are $\sin\frac{\pi-A}{2}, \sin\frac{\pi-B}{2}, \sin\frac{\pi-C}{2}$, and that the angles $(\pi-A)/2$, etc., are positive and add to 180°. Since in any triangle DEF of sides d, e, f, we have $d = 2R\sin D$, etc., where R is the circumradius, it follows that $\sin\frac{\pi-A}{2}, \sin\frac{\pi-B}{2}$, $\sin\frac{\pi-C}{2}$ are sides of a triangle with *fixed* circumradius $1/2$. It is well known and easy to prove that of all triangles inscribed in a given circle, the equilateral triangle has the maximum area. Thus, our maximum area corresponds to an equilateral triangle of sides $\cos 30°$, or $3\sqrt{3}/16$.

As a bonus, by using the known result that the product of the sides of a triangle equals four times the product of its area and circumradius, we have the triangle identity

$$\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} = 4s\left(s-\cos\frac{A}{2}\right)\left(s-\cos\frac{B}{2}\right)\left(s-\cos\frac{C}{2}\right).$$

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Gleason, Andrew M. (ed.), Proceedings of the International Congress of Mathematicians, August 3-11, 1986. 2 vols., American Mathematical Society, 1987; cii + 1708 pp.

The International Congresses of Mathematicians provide an opportunity for succinct summaries of the state of the subdisciplines of mathematics. Sometimes a summary is either of interest or understandable only to specialists. Any mathematician, however, will find some worthwhile items in this collection of addresses, such as De Branges on the Bieberbach conjecture, Donaldson on 4-manifolds, and Lenstra on factoring integers via elliptic curves. A number of addresses concern computation: the complexity of equation solving (Schoenhage, Smale), antidifferentiation (Kechris), polynomial algebra (Grigor'ev), and data retrieval and compression (Krichevsky), as well as "How to prove a theorem so no one else can claim it" (zero-knowledge proofs, Blum) and "The search for provably secure identification schemes" (Shamir). Perhaps the most remarkable address is the one that all your students should read: "The centrality of mathematics in the history of Western thought" (Grabiner). [The Grabiner article will appear in an upcoming issue of *Mathematics Magazine*. Ed.]

Grattan-Guinness, I. (ed.), *History in Mathematics Education: Proceedings of a Workshop Held at the University of Toronto, Canada, July-August* 1983, Belin (8, rue Frou, 75006 Paris), 1987; 208 pp (P).

Collection of essays related to the place and use of history in mathematics education. Particularly notable is the essay on "Biography in the mathematics classroom" (Pycior).

Klamkin, Murray (ed.), *Mathematical Modelling: Classroom Notes in Applied Mathematics.* SIAM, 1987; xiv + 338 pp (P).

Collection of columns from SIAM Review since 1975, classified into physical/ mathematical sciences and life sciences (no Notes have ever been submitted in the behavioral sciences!). The editor has added valuable substantial collections of supplementary references. This collection provides a tremendous resource for undergraduate courses in mathematical modeling; it also demonstrates the importance of physics, geometry, and probability in the education of mathematical modelers.

Keen, Linda (ed.), *The Legacy of Sonya Kovalevskaya: Proceedings of a Symposium Sponsored by the Association for Women in Mathematics and the Mary Ingraham Bunting Institute Held October* 25-28, 1985, AMS, 1987; xii + 297 pp, $29 (P).

In addition to research papers connected to the work of Kovalevskaya, this volume contains a biographical sketch (Koblitz), an assessment of her impact on her contemporaries (Cooke), and an account of the "systematic mythification" after her death that both "diminished her perceived importance" and "obscured her value as a role model" (Koblitz). Two of the research papers, on quasicrystals, are expository in flavor and will interest the general reader.

Gardner, Martin, *Riddles of the Sphinx and Other Mathematical Puzzle Tales*, MAA, 1987; x + 164 pp, (P).

Collection of the third (and final) collection of puzzle columns contributed by Gardner to Isaac Asimov's *Science Fiction Magazine.* Each problem comes with an answer, which in turn raises another puzzle, whose solution may suggest a third puzzle; in some instances, there is even a fourth!

Schattschneider, Doris, and Wallace Walker, *M. C. Escher Kaleidocycles*, rev. ed., Pomegranate Artbooks, 1987; 57 pp (P) + 17 full-color models, $13.95.

Splendid reissue of a geometric classic of our times, in which three-dimensional rings of tetrahedra exhibit on their surfaces repeating patterns of Escher's. Completely redesigned edition, with revised text and illustrations, and with remade scoring and cutting dies for the models.

Weingartner, Paul, and Leopold Schmetterer (eds.), *Gödel Remembered*, Salzburg 10-12 July 1983, Humanities Press, 1987; 187 pp, $65.

Essays by R. Gödel (brother of K. Gödel) (on the history of the family), O. Taussky-Todd (remembrances from Vienna and later), S. C. Kleene (impressions on students of logic in the 1930s), and G. Kreisel (Gödel on intuitionistic logic). In addition, a previously-unpublished letter of Zermelo's is reproduced; it shows him to have remained unconvinced of Gödel's achievements after the two had met in 1931. (Note: Kreisel's essay takes up two-thirds of the book.)

Wang, Hao, *Reflections on Kurt Gödel*, MIT Press, 1987; xxvi + 336 pp, $25.

Landmark biography of a reclusive genius, by a colleague who knew both the man and his work well. Without negelecting Gödel's mathematical achievements, Wang emphasizes Gödel's philosophical side.

Bellin, David, and Gary Chapman (eds.), *Computers in Battle: Will They Work?*, Harcourt Brace Jovanovich, 1987; xiv + 362 pp, $14.95.

As a teacher of computer science I have always expressed skepticism about the reliability of large software projects. This year I have also been supervising campus microcomputer maintenance and repair, and I am overwhelmingly impressed with the "flakiness" of the hardware. But my experience is subjective and anecdotal. This book advances scientific, technological, and philosophical arguments why we should not trust computers in battle. Those who have pledged their allegiance to Star Wars or their faith to the perfectability of technology are unlikely to be convinced, while those already skeptical of computers will feel reinforced; those with open minds may benefit the most.

Edelstein-Keshet, Leah, *Mathematical Models in Biology*, Random House, 1988; xviii + 586 pp, $32.

Considers discrete, continuous, and multivariable models, but not stochastic ones. Calculus is a prerequisite; although linear algebra and differential equations topics are explained as needed, a student without background in those subjects is going to feel ill-at-ease and not derive as much benefit. Full benefit will accrue only to those who also consult the many references cited and thereby assess the practical usefulness of the models developed in the text.

Bottazzini, Umberto, *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*, Springer-Verlag, 1986; 332 pp.

Valuable historical perspectives on analysis, suitable for students taking advanced calculus or analysis. (Unfortunately, this book is printed in one of the more ugly combinations of laser-printed typefaces, especially the mathematical expressions.)

Francis, George K., *A Topological Picturebook*, Springer-Verlag, 1987; xv + 194 pp.

"My book is about how to draw mathematical pictures . . . You should regard this book as a description of my own graphical calculus." The author offers instruction in how to draw by hand accurate and informative figures. Moreover, each section is accompanied by a "picture story" about the underlying mathematics.

Ashenhurst, Robert L., and Susan Graham, *ACM Turing Award Lectures: The First Twenty Years* 1966-1985, ACM, 1987; xviii + 483 pp.

Collection of lectures by individuals "selected for contributions of a technical nature to the mathematical community." There is much here of enduring value, by the great people of the discipline. Subsequent Turing lectures have appeared in the March issues of *Communications of the ACM*, and John McCarthy's 1971 lecture finally was published in the December 1987 issue.

Peterson, Ivars, A shortage of small numbers, *Science News* 133 (9 January 1988) 31.

"There aren't enough small numbers to meet the many demands made of them." That's Richard Guy's (Calgary) Strong Law of Small Numbers. What he is referring to is the annoying habit of patterns to hold for small n but fail further on: in some sense, the pattern is "simply a figment of the smallness of the values of n for which the example has been worked out."

Peterson, I., The curious power of large numbers, *Science News* 133 (30 January 1988) 70.

N. D. Elkies (Harvard) has found three fourth powers whose sum is another fourth power, a counterexample to a conjecture of Euler's in 1769.

Douglas, Ronald G., Today's calculus courses are too watered down and outdated to capture the interest of students, *Chronicle of Higher Education* (20 January 1988) B1, B3.

"About all that students learn in a typical undergraduate course . . . is to pass tests by solving problems just like the ones in the textbook, except that the numbers are different. Word problems have largely disappeared . . . If . . . background from economics or physics or biology is presented along with an equation, it only gets in the way of what students must learn to pass the tests. That calculus had any role in setting up the equation is lost on the students, as is the way one would use calculus if the equation had to be changed. . . . Calculus is one of the few college subjects in which the instructor assumes that students have learned the necessary basics in high school. . . . Failure in calculus causes more career changes than failure in any other course. . . . If we want to ensure that more Americans qualify for science and science-related careers, the way we teach calculus will have to change."

Knorr, Wilbur, *The Ancient Tradition of Greek Problems*, Birkhäuser, 1986; ix + 411 pp, $69.

Much as Hilbert's famous 23 problems have led to much mathematics of the 20th century, the three "classical problems" of cube duplication, angle trisection, and circle quadrature fueled much of ancient mathematics. Knorr investigates the origins of the interest in these problems, progress on them, and assessment by the ancients themselves of the degrees of success.

Hazewinkel, M., *Encyclopaedia of Mathematics*, Vol. 1: A-B, Reidel, 1988; ix + 488 pp.

An updated and annotated translation of the Soviet Mathematical Encyclopaedia (5 vols., 1977-1985). This is the first of 10 vols., including an index volume. Includes survey articles, shorter articles on narrower topics, and short definitions. All articles have been classified according to the 1980 AMS scheme, and the index volume will include an index by classification number. The added annotations add substantially to the value of the encyclopedia, which should occupy a prominent place in every mathematics library and departmental common room.

Peters, William S., *Counting for Something: Statistical Principles and Personalities*, Springer-Verlag, 1987; xviii + 275 pp.

Suitable as a supplement to the usual elementary statistics course, this book is "more a 'show-and-tell' account than a 'how-to-do-it' manual." Instructors will find it useful for enrichment material, and students will find it interesting to read.

Frontiers of Chaos, distr. by Computer Art Resource (Box 2069, Mill Valley, CA 94942); 30-min. videotape (VHS or Beta), $30.

Ten pans and zooms around the Mandelbrot set, set to synthesized music. The motion brings the fractals greater life than still photos, for an utterly impressive artistic effect.

Nothing but Zooms, distr. by Art Matrix (Box 880, Ithaca, NY 14851-0880); 90-min. videotape (VHS or Beta), $50.

Zooms into the Mandelbrot set, "Julia promenades," and other amazing scenes, representing more than 100 hours of supercomputer time. Amazing, astonishing, alluring.

NEWS & LETTERS

48TH PUTNAM COMPETITION: WINNERS AND SOLUTIONS

Teams from 277 schools competed in the 1987 William Lowell Putnam Mathematical Competition. The top five winning teams, in descending rank, are:

Harvard University
David J. Moews, Bjorn M. Poonen, Michael Reid
Princeton University
Daniel J. Bernstein, David J. Grabiner, Matthew D. Mullin
Carnegie Mellon University
Petros I. Hadjicostas, Joseph G. Keane, Karl M. Westerberg
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Solutions to the 1987 *Putnam problems were prepared for publication in this* Magazine *by Loren Larson, St. Olaf College.*

A-1. Curves A, B, C, and D, are defined in the plane as follows:

$$A = \left\{(x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\},$$

$$B = \left\{(x, y) : 2xy + \frac{x}{x^2 + y^2} = 3\right\},$$

$$C = \left\{(x, y) : x^3 - 3xy^2 + 3y = 1\right\},$$

$$D = \left\{(x, y) : 3x^2y - 3x - y^3 = 0\right\}.$$

Prove that $A \cap B = C \cap D$.

Sol. First note that (0, 0) doesn't belong to either $A \cap B$ or $C \cap D$, so in what follows suppose that $(x,y) \neq (0, 0)$.

Let Eq(i), $i = 1,2,3,4$, denote the equation that defined the set A, B, C, D, respectively. Also, let $f(x, y)$Eq(i) denote the equation obtained by multiplying each side of Eq(i) by $f(x, y)$.

The matrix product

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} \text{Eq}(1) \\ \text{Eq}(2) \end{pmatrix} = \begin{pmatrix} \text{Eq}(3) \\ \text{Eq}(4) \end{pmatrix}$$

shows that $A \cap B \subseteq C \cap D$, and

$$\begin{pmatrix} \text{Eq}(1) \\ \text{Eq}(2) \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}\begin{pmatrix} \text{Eq}(3) \\ \text{Eq}(4) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x}{x^2 + y^2} & \frac{x}{x^2 + y^2} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{pmatrix}\begin{pmatrix} \text{Eq}(3) \\ \text{Eq}(4) \end{pmatrix}$$

shows that $C \cap D \subseteq A \cap B$.

These two inclusions show that $A \cap B = C \cap D$.

A-2. The sequence of digits

1 2 3 4 5 6 7 8 9 1 0 1 1 1 2 1 3 1 4 1 5 1 6 1 7 1 8 1 9 2 0 2 1 $\cdots$

is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

Sol. The r-digit numbers run from 10^{r-1} to $10^r - 1$, so there are $10^r - 10^{r-1}$ of them. Thus, the total number of digits in numbers with at most r digits is $g(r) = \sum_{k=1}^{r} k(10^k - 10^{k-1}) = -1 + \sum_{k=1}^{r-1}(k - (k + 1))10^k + r10^r = - \sum_{k=0}^{r-1} 10^k + r10^r = r10^r - \frac{10^r - 1}{9}$ for $r \geq 1$. But $0 < \frac{10^r - 1}{9} < 10^r$, so $(r - 1)10^r < g(r) < r10^r$. Thus, $g(1983) < 1983 \cdot 10^{1983} < 10^4 \cdot 10^{1983} = 10^{1987}$, and $g(1984) > 1983 \cdot 10^{1984} > 10^3 \cdot 10^{1984} = 10^{1987}$. It follows that $f(1987) = 1984$.

A-3. For all real x, the real valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x .$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.
(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

Sol. The general solution to the differential equation is $f(x) = (x^2 + bx + c)e^x$ with b and c real. For such a function $f'(x) = (x^2 + (b+2)x + (b+c))e^x$. Clearly $f(x) > 0$ for all x if and only if $D = b^2 - 4ac < 0$, and $f'(x) > 0$ for all x if and only if $D' = (b+2)^2 - 4(b+c) = b^2 + 4b + 4 - 4b - 4c = D + 4 < 0$. The answer to (a) is "no" because $D < 0$ does not imply $D' < 0$. (For example, take $b = c = 1$; then $f'(x) > 0$ for all x but $f'(1) = 0$.) The answer to (b) is "yes" because $D' < 0$ implies $D < 0$.

A-4. Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2F(y - x, z - x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1, 0, 0) = 4$, $P(0, 1, 0) = 5$, and $P(0, 0, 1) = 6$. Also let A, B, C be complex numbers with $P(A, B, C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

Sol. Letting $u = 1$ and $x = 0$, we have $F(y,z) = P(0,y,z)$ is a polynomial. $F(uy,uz) = P(0,uy,uz) = u^2F(y,z)$, so F is homogeneous of degree 2. Now $P(x,y,z) = F(y-x, z-x)$ implies that

$$P(x,y,z) = a(y-x)^2 + b(y-x)(z-x) + c(z - x)^2$$

with a,b,c real. Then $4 = P(1,0,0) = a + b + c$, $5 = P(0,1,0) = a$, and $6 = P(0,0,1) = c$. It follows that $4 = a + b + c = 5 + b + 6$, and so $b = -7$.

For the complex numbers A,B,C of the hypothesis, we have $5(B - A)^2 - 7(B - A)(C - A) + 6(C - A)^2 = 0$. Let $m = \frac{C - A}{B - A}$. Then $5 - 7m + 6m^2 = 0$. The roots of $6m^2 - 7m + 5 = 0$ are complex, so $|m| = \sqrt{\frac{5}{6}}$. Hence, $|C - A| = \sqrt{\frac{5}{6}}\,|B - A| = \frac{5}{3}\sqrt{30}$.

A-5. Let $\vec{G}(x,y) = \left(\frac{-y}{x^2 + 4y^2}, \frac{x}{x^2 + 4y^2}, 0\right)$.

Prove or disprove that there is a vector-valued function

$$\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties:
(i) M, N, P have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;
(ii) Curl $\vec{F} = \vec{0}$ for all $(x,y,z) \neq (0,0,0)$;
(iii) $\vec{F}(x,y,0) = \vec{G}(x,y)$.

Sol. Note that Curl $\vec{G} = \vec{0}$ unless (x,y,z) is on the z-axis. If $\vec{F}$ exists, then by Stokes' Theorem,

$$\int_C \vec{G}(x,y)\cdot d\vec{r} = \int_C \vec{F}(x,y,z)\cdot d\vec{r} = \iint_S (\text{Curl } \vec{F})\cdot \vec{n}\, dS = 0$$

where C is the ellipse $x^2 + 4y^2 = 1$, $z = 0$, in the xy-plane and S is the part of the ellipsoid $x^2 + 4y^2 + z^2 = 1$ with $z \geq 0$. However, the integral is not zero. For example, on C, $x^2 + 4y^2 = 1$ and thus,

$$\int_C \vec{G}(x,y)\cdot d\vec{r} = \int_C (-y\,\vec{i} + x\,\vec{j})\cdot d\vec{r} = \iint_E 2\, dx\, dy = 2 \text{ Area}(E)$$

where E is the interior of C. Thus $\vec{F}$ does not exist.

A-6. For each positive integer n, let $a(n)$ be the number of zeros in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

Sol. For each integer $k \geq 0$, the integer n in base 3 has $k+1$ digits iff $3^k \leq n < 3^{k+1} - 1$. Among the integers in this interval there are $\binom{k}{i}2^{k+1-i}$ for which $a(n) = i$, so $\sum_{n=3^k}^{3^{k+1}-1} x^{a(n)} = \sum_{i=0}^{k} \binom{k}{i}x^i 2^{k+1-i} = 2(x+2)^k$. Thus

$$\frac{2(x+2)^k}{3^{3k+3}} < \sum_{n=3^k}^{3^{k+1}-1} \frac{x^{a(n)}}{n^3} < \frac{2(x+2)^k}{3^{3k}},$$

and therefore

$$\frac{2}{27}\sum_{k=0}^{m}\left(\frac{x+2}{27}\right)^k < \sum_{n=1}^{3^{m+1}-1}\frac{x^{a(n)}}{n^3} < 2\sum_{k=0}^{m}\left(\frac{x+2}{27}\right)^k.$$

It follows that the series converges (for $x > 0$) iff $\frac{x+2}{27} < 1$; that is, $0 < x < 25$.

B-1. Evaluate $\int_2^4 \frac{\sqrt{\ln (9 - x)}\, dx}{\sqrt{\ln (9 - x)} + \sqrt{\ln (x + 3)}}$.

Sol. Let I denote the value of the integral. The substitution $9 - x = y + 3$ gives

$$I = \int_2^4 \frac{\sqrt{\ln (y + 3)}\, dy}{\sqrt{\ln (y + 3)} + \sqrt{\ln (9 - y)}}, \text{ so}$$

$$2I = \int_2^4 \frac{\sqrt{\ln (x + 3)} + \sqrt{\ln (9 - x)}}{\sqrt{\ln (x + 3)} + \sqrt{\ln (9 - x)}}\, dx = 2,$$

and $I = 1$.

B-2. Let r, s, and t be integers with $0 \le r$, $0 \le s$, and $r + s \le t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \frac{\binom{s}{2}}{\binom{t}{r+2}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t + 1 - s)\binom{t-s}{r}}.$$

(Note: $\binom{n}{k}$ denotes the binomial coefficient

$$\frac{n(n - 1) \cdots (n + 1 - k)}{k(k - 1) \cdots 3 \cdot 2 \cdot 1}.)$$

Sol. Let

$$F(r,s,t) = \frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}}.$$

Then

$$F(r,s,t) = \frac{\binom{s-1}{0}}{\binom{t}{r}} + \frac{\binom{s-1}{0} + \binom{s-1}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s-1}{s-2} + \binom{s-1}{s-1}}{\binom{t}{r+s-1}} + \frac{\binom{s-1}{s-1}}{\binom{t}{r+s}}$$

$$= F(r, s - 1, t) + F(r + 1, s - 1, t).$$

The proof now follows easily by induction on s.

B-3. Let F be a field in which $1 + 1 \ne 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x, y) = (1, 0)$ and $(x, y) = \left(\frac{r^2 - 1}{r^2 + 1}, \frac{2r}{r^2 + 1}\right)$, where r runs through the elements of F such that $r^2 \ne -1$.

Sol. Let $x_r = \frac{r^2 - 1}{r^2 + 1}$ and $y_r = \frac{2r}{r^2 + 1}$ for r any element of F such that $r^2 \ne -1$.

It is easy to check that $(1,0)$ and (x_r,y_r) satisfy $x^2 + y^2 = 1$.

The problem is thus to show that if x, y are in F with $(x,y) \ne (1,0)$ and $x^2 + y^2 = 1$, then $x = x_r$ and $y = y_r$ for some r. We observe that for $x_r \ne 1$, $r = \frac{y_r}{1 - x_r}$, and this suggests that we set $r = \frac{y}{1 - x}$. Then we have

$$r^2 + 1 = \left(\frac{y}{1 - x}\right)^2 + 1 = \frac{y^2 + (1 - x)^2}{(1 - x)^2}$$

$$= \frac{y^2 + x^2 - 2x + 1}{(1 - x)^2} = \frac{2 - 2x}{(1 - x)^2} = \frac{2}{1 - x} \ne 0,$$

and therefore $r^2 - 1 = \frac{2}{1 - x} - 2 = \frac{2x}{1 - x}$. It follows that $x_r = \frac{r^2 - 1}{r^2 + 1} = x$ and $y_r = \frac{2r}{r^2 + 1} = y$.

B-4. Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \cdots$ For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

Sol. Let $y_0 = \text{Arccos } 0.8$ and $\theta_n = y_0 + y_1 + \cdots + y_n$. Then $x_{n+1} = \cos \theta_n$ and $y_{n+1} = \sin \theta_n$. Since $\sin \theta = \sin(\pi - \theta) \le \theta - \pi$ for $0 \le \theta \le \pi$, one easily proves by induction that $0 < \theta_n \le \theta_{n+1} \le \pi$ for $n = 1, 2, 3, \ldots$. Hence $L = \lim_{n\to\infty} \theta_n$ exists since $\theta_0, \theta_1, \ldots$ is a monotonic bounded sequence. It follows that $\lim_{n\to\infty} y_n = 0$.

Since $\cos t$ and $\sin t$ are continuous for all real

t, $0 = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} \sin \theta_n = \sin(\lim_{n \to \infty} \theta_n) = \sin L$. As $0 < L \leq \pi$, this implies that $L = \pi$. Then $\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \cos \theta_n = \cos(\lim_{n \to \infty} \theta_n) = \cos L = \cos \pi = -1$.

B-5. Let O_n be the n-dimensional zero vector $(0, 0, \cdots, 0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex z_t, not all zero, then at least one of the z_t is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots w_n$ such that

$$\operatorname{Re}\left[M\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right] = \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}$$

(Note: If C is a matrix of complex numbers, $\operatorname{Re}(C)$ is the matrix whose entries are the real parts of entries of C.)

Sol. Write $M = A + iB$ where A and B are real $2n \times n$ matrices and $N = (A\ B)$, a real $2n \times 2n$ matrix. If $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = u + iv$ where u and v are real column vectors of length n, then $\operatorname{Re}(Mw) = \operatorname{Re}[(A + iB)(u + iv)] = Au - Bv = (A\ B)\begin{pmatrix} u \\ -v \end{pmatrix} = N\begin{pmatrix} u \\ -v \end{pmatrix}$ and so we need to show that N is invertible. Suppose that $x = (x_1, \ldots, x_{2n})$ is a real vector of length $2n$ such that $xN = O_{2n}$. Then $xA = O_n$, $xB = O_n$ and hence $xM = x(A + iB) = O_n$. Therefore, by hypothesis, $x = O_{2n}$ and hence N has an inverse.

B-6. Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $\frac{p^2 - 1}{2}$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.

Sol. For a in S, let $2a = \epsilon_a s_a$ where s_a is in S and $\epsilon_a = \pm 1$. Set $M = \frac{p^2 - 1}{2} - N$ so that M is the number of a in S such that $\epsilon_a = -1$. If a and b are in S and $s_a = s_b$ then $a = \pm b$ and hence $a = b$. Therefore, as a runs through S, s_a runs through S as well. Hence in F,

$$2^{(p^2 - 1)/2} \prod_{a \in S} a = \prod_{a \in S} (\epsilon_a s_a) = (-1)^M \prod_{a \in S} s_a$$
$$= (-1)^M \prod_{a \in S} a.$$

Hence, $(-1)^M = 2^{(p^2 - 1)/2}$ Using Lagrange's Theorem for finite groups or the Euler Theorem or the little Fermat Theorem, one has $2^{(p^2 - 1)/2} = (2^{p-1})^{(p+1)/2} = (1)^{(p+1)/2} = 1$ and therefore M is even. But $\frac{p^2 - 1}{2}$ is also even and therefore N is even.

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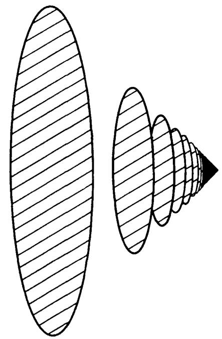

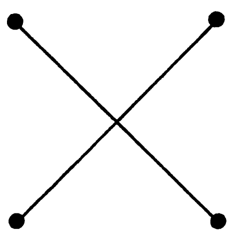

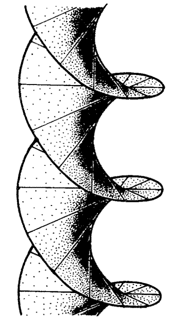

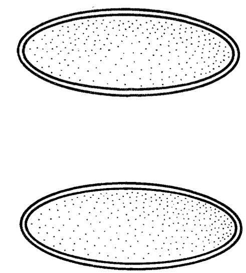

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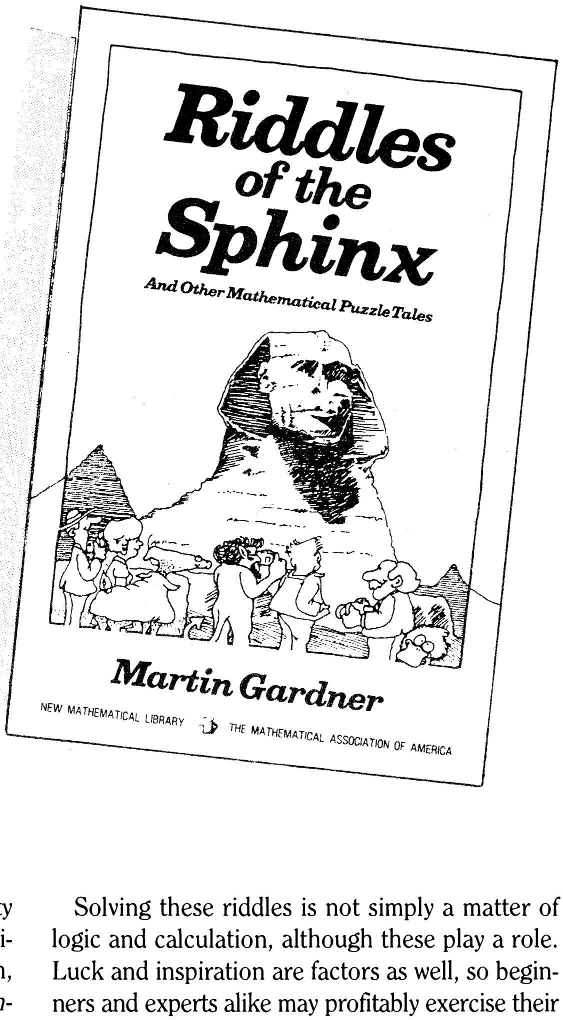

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Martin Gardner has charmed readers for over fifty years with his delightful books and articles on science. He is best known for the popular column, *"Mathematical Games,"* which appeared in *Scientific American* for twenty-five years. Generations of scientists and mathematicians have been inspired by his writing and the MAA is proud to include his name in its list of authors.

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